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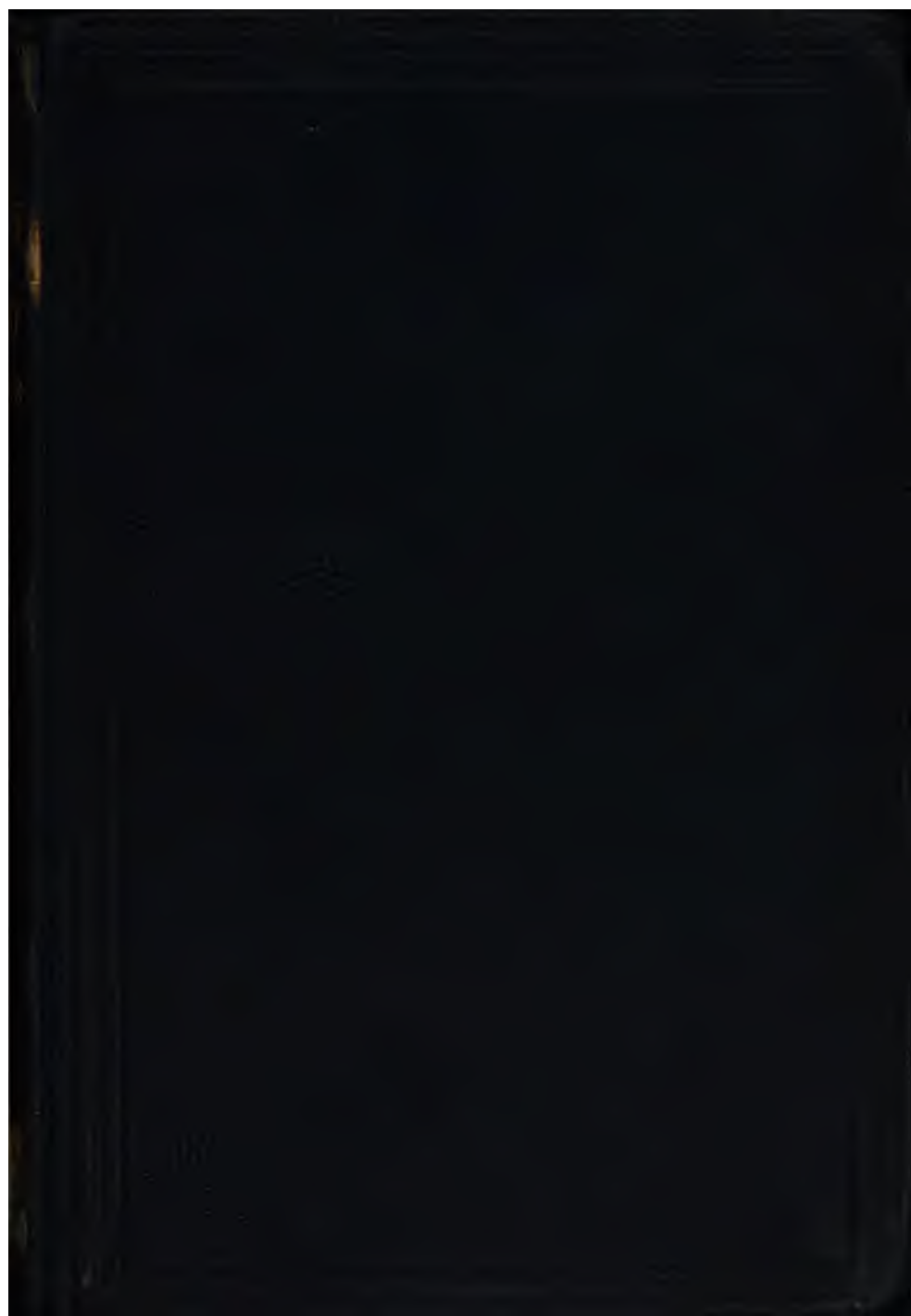
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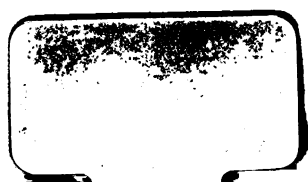
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**KEY**

**TO**

**EXERCISES IN EUCLID.**



KEY  
TO  
EXERCISES IN EUCLID.

BY  
I. TODHUNTER, M.A., F.R.S.,  
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London :  
MACMILLAN AND CO.  
1880.

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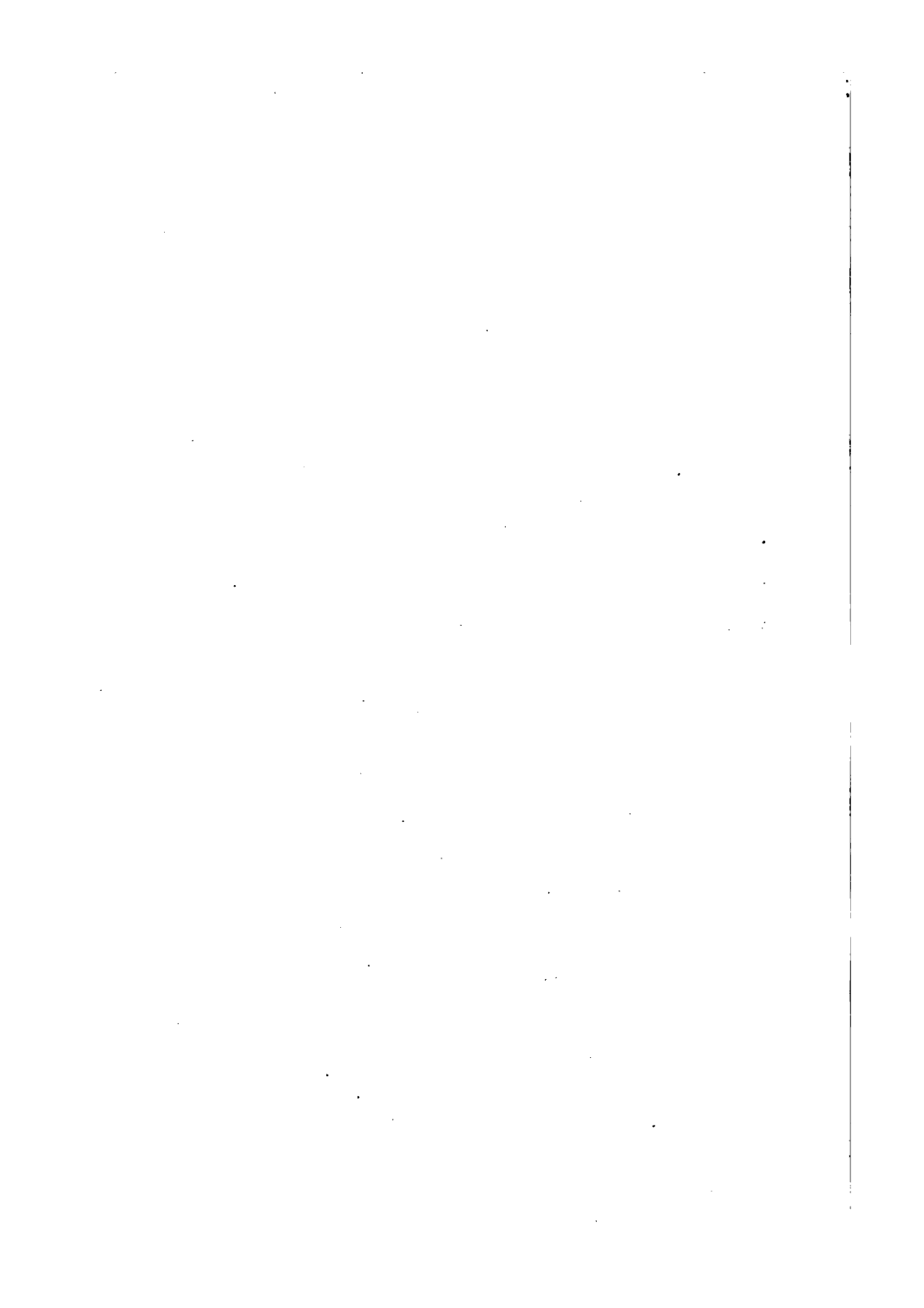


**Cambridge:**

**PRINTED BY C. J. CLAY, M.A.  
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THE Keys already issued to some of the Author's works have been found very useful by affording assistance to private students, and by saving the labour and time of teachers; and this has led to the issue of the present volume. Care has been taken, as in the former Keys, to present the solutions in a simple natural manner, in order to meet the difficulties which are most likely to arise, and to render the work intelligible and instructive.

*November, 1880.*



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## EXERCISES IN EUCLID.

### I. 1 to 15.

1. Let  $AB$  be the given straight line on which the isosceles triangle is to be constructed; let  $DE$  be the straight line to which each side is to be equal. With centre  $A$ , and radius equal to  $DE$ , describe a circle; with centre  $B$ , and radius equal to  $DE$ , describe another circle; let these circles intersect at  $C$ . Join  $AC$  and  $BC$ ; then  $ABC$  will be the triangle required.

2. The given point and the vertex of the constructed triangle both fall on the circumference of the small circle.

3. Let  $AB$  and  $CD$  be two straight lines which bisect each other at right angles at the point  $O$ ; so that  $AO$  is equal to  $OB$ ,  $CO$  is equal to  $OD$ , and the angles at  $O$  are right angles. In  $CD$  take any point  $E$ , and join  $EA$  and  $EB$ ; then  $EA$  shall be equal to  $EB$ .

For  $AO$  is equal to  $BO$  by hypothesis;  $EO$  is common to the two triangles  $AOE$  and  $BOE$ ; and the angle  $AOE$  is equal to the angle  $BOE$  by Axiom 11. Therefore  $EA$  is equal to  $EB$ , by I. 4.

Similarly it may be shewn that any point in  $AB$  is equally distant from  $C$  and  $D$ .

4. The angles  $ABC$  and  $ACB$  are equal by I. 5. Hence the angles  $DBC$  and  $DCB$  are equal by Axiom 7. Therefore the sides  $DB$  and  $DC$  are equal by I. 6.

5. The angle  $DBA$  is half the angle  $ABC$ , by construction. The angle  $BAD$  is equal to half the angle  $ABC$ , by hypothesis. Therefore the angle  $DBA$  is equal to the angle  $BAD$ . Therefore  $BD$  is equal to  $AD$  by I. 6.

6. It is shewn in the demonstration of I. 5, that the angle  $BCF$  is equal to the angle  $CBG$ ; therefore  $BH$  is equal to  $CH$ , by I. 6. Also it is shewn that  $FC$  is equal to  $GB$ . Therefore  $FH$  is equal to  $GH$  by Axiom 3.

7.  $AF$  is equal to  $AG$ , by construction;  $AH$  is common to the two triangles  $FAH$  and  $GAH$ ; and  $FH$  is equal to  $GH$ , by Exercise 6: therefore the angle  $FAH$  is equal to the angle  $GAH$  by I. 8.

8.  $AB$  is equal to  $AD$ , by hypothesis;  $AC$  is common to the two triangles  $BAC$ ,  $DAC$ ; and the angle  $BAC$  is equal to the angle  $DAC$ , by hypothesis: therefore the base  $BC$  is equal to the base  $DC$ , and the angle  $ACB$  is equal to the angle  $ACD$  by I. 4.

9. The angle  $ACB$  is equal to the angle  $BDA$  by I. 8; and then the two triangles  $ACB$  and  $BDA$  are equal in all respects by I. 4; so that the angle  $ABC$  is equal to the angle  $BAD$ . Therefore  $AO$  is equal to  $BO$  by I. 6.

10. Let  $ABCD$  be the rhombus, so that  $AB, BC, CD, DA$  are all equal. Join  $BD$ . Then in the two triangles  $BAD, BCD$  the base  $BD$  is common; and the two sides  $BA, AD$  are equal to the two sides  $BC, CD$ , each to each; therefore the angle  $BAD$  is equal to the angle  $BCD$ , by I. 8. Similarly it may be shewn that the angle  $ABC$  is equal to the angle  $ADC$ .

11. Let  $ABCD$  be the rhombus, so that  $AB, BC, CD, DA$ , are all equal. Join  $BD$ . Then in the two triangles  $ABD, CBD$ , the side  $AB$  is equal to the side  $CB$ , the side  $BD$  is common, and the base  $AD$  is equal to the base  $CD$ ; therefore the angle  $ABD$  is equal to the angle  $CBD$  by I. 8. Thus the angle  $ABC$  is bisected by  $BD$ . Similarly it may be shewn that the angle  $ADC$  is bisected by  $BD$ ; and also that the angles  $BAD$  and  $BCD$  are bisected by  $AC$ .

12. Let there be two isosceles triangles  $ACB, ADB$  on the same base  $AB$ , and on *opposite* sides of it. Join  $CD$ ; then  $CD$  shall bisect  $AB$  at right angles.

In the two triangles  $ACD, BCD$  the side  $CD$  is common;  $AC$  is equal to  $BC$ , by hypothesis; and the base  $AD$  is equal to the base  $BD$ , by hypothesis; therefore the angle  $ACD$  is equal to the angle  $BCD$  by I. 8.

Let  $AB$  and  $CD$  intersect at  $E$ . Then in the triangles  $ACE, BCE$  the side  $CE$  is common; the side  $AC$  is equal to the side  $BC$ ; and the angle  $ACE$  has been shewn equal to the angle  $BCE$ : therefore the triangles are equal in all respects by I. 4. Thus  $AE$  is equal to  $BE$ ; and the angle  $AEC$  is equal to the angle  $BEC$ , so that each of them is a right angle.

Next let the two isosceles triangles  $ACB, ADB$  be on the same base  $AB$ , and on the *same* side of it. Join  $CD$  and produce it to meet  $AB$  at  $E$ . It may be shewn as before that  $AE$  is equal to  $BE$ , and that the angles at  $E$  are right angles.

13. Let  $AB$  be the given straight line,  $C$  and  $D$  the two given points. Join  $CD$  and bisect it at  $E$ . From  $E$  draw a straight line at right angles to  $CD$ , meeting  $AB$  at  $F$ . Join  $CF, DF$ . Then  $CF$  shall be equal to  $DF$ .

For in the two triangles  $CEF$  and  $DEF$ , the side  $EF$  is common,  $CF$  is equal to  $DE$ , and the right angle  $CEF$  is equal to the right angle  $DEF$ : therefore  $CF$  is equal to  $DF$ , by I. 4.

The problem is impossible when the two points  $C$  and  $D$  are situated on the same perpendicular to the given straight line  $AB$ , and at unequal distances from that straight line.

14. Let  $AB$  be the given straight line,  $C$  and  $D$  the two given points. From  $D$  draw  $DE$  perpendicular to  $AB$ , and produce  $DE$  through  $E$  to a point  $F$  such that  $EF=ED$ . Join  $CF$ , and produce it to meet  $AB$  at  $G$ ; join  $DG$ : then  $CG$  and  $DG$  shall be the required straight lines.

For  $ED$  is equal to  $EF$ , and  $EG$  is common to the two triangles  $EDG$  and  $EFG$ ; the right angles  $GED$  and  $GEF$  are equal: therefore by I. 4 the triangles  $FEG$  and  $DEG$  are equal in all respects, so that the angle  $FGE$  is equal to the angle  $DGE$ .

The problem is impossible when the two points  $C$  and  $D$  are equally distant from the straight line  $AB$ , and not on the same perpendicular to  $AB$ .

If  $C$  and  $D$  are equally distant from  $AB$ , and on the same perpendicular, then any point in  $AB$  may be taken for the point  $G$ .

15. Let the angle  $BAC$  be bisected by  $AD$ , and the angle  $BAG$  by  $AE$ : the angle  $DAE$  shall be a right angle.

Since the angle  $BAD$  is half the angle  $BAC$ , and the angle  $BAE$  is half the angle  $BAG$ , the two angles  $BAD$  and  $BAE$  together are half the two angles  $BAC$  and  $BAG$  together. But the angles  $BAC$  and  $BAG$  together are equal to two right angles, by I. 13; therefore the angles  $BAD$  and  $BAE$  together are equal to a right angle.

16. Let the four straight lines  $AE$ ,  $BE$ ,  $CE$ ,  $DE$  meet at the point  $E$ , and make the angle  $AEB$  equal to the angle  $CED$ , and the angle  $BEC$  equal to the angle  $DEA$ : then shall  $AE$  and  $EC$  be in one straight line, and also  $BE$  and  $ED$  in one straight line.

By I. 15, Cor. 2 the four angles  $AEB$ ,  $BEC$ ,  $CED$ ,  $DEA$  are together equal to four right angles; but the two angles  $AEB$  and  $BEC$  are equal to the two angles  $CED$  and  $DEA$ ; therefore the angles  $AEB$  and  $BEC$  are together equal to two right angles; therefore  $AE$  and  $EC$  are in one straight line by I. 14. Similarly it may be shewn that  $BE$  and  $ED$  are in one straight line.

#### I. 16 to 26.

17. The angle  $BDA$  is greater than the angle  $CAD$ , by I. 16; the angle  $CAD$  is equal to the angle  $BAD$ , by hypothesis: therefore the angle  $BDA$  is greater than the angle  $BAD$ . Therefore the side  $BA$  is greater than the side  $BD$ , by I. 19. Similarly it may be shewn that  $CA$  is greater than  $CD$ .

18. Take any point  $G$  in  $BC$ , and join  $AG$ . The angle  $AGC$  is greater than the angle  $ABC$ , by I. 16; and the angle  $AGB$  is greater than the angle  $ACB$ , by I. 16. Therefore the angles  $ABC$  and  $ACB$  are together less than the angles  $AGC$  and  $AGB$  together; therefore the angles  $ABC$  and  $ACB$  are together less than two right angles by I. 13.

19. Join  $BD$ . The angle  $ABD$  is greater than the angle  $ADB$ , and the angle  $DBC$  is greater than the angle  $BDC$ , by I. 18; therefore the whole angle  $ABC$  is greater than the whole angle  $ADC$ . Similarly by joining  $AC$  we can shew that the angle  $DCB$  is greater than the angle  $DAB$ .

20. Let  $ABCD$  be the square; on  $BC$  take any point  $E$ ; join  $AE$  and produce it to meet  $DC$  produced at  $F$ : then shall  $AF$  be greater than  $AC$ .

The angle  $DCA$  is greater than the angle  $CFA$  by I. 16. The angle  $ACF$ , which is greater than a right angle, is greater than the angle  $DCA$ , which is less than a right angle. Therefore the angle  $ACF$  is greater than the angle  $AFC$ . Therefore  $AF$  is greater than  $AC$ , by I. 19.

21. Let  $AB$  be the given straight line,  $O$  the given point without it. From  $O$  draw  $OC$  perpendicular to  $AB$ ; then  $OC$  shall be shorter than any other straight line  $OD$  drawn from  $O$  to  $AB$ .

For the angle  $OCD$  is a right angle; therefore the angle  $ODC$  is less than a right angle, by I. 17: therefore  $OC$  is less than  $OD$ , by I. 19.

Next, let  $OE$  be a straight line drawn from  $O$  to  $AB$ , and more remote from  $OC$  than  $OD$  is:  $OD$  shall be less than  $OE$ .



For the angle  $ODC$  is greater than the angle  $OEC$ , by I. 16; and the obtuse angle  $ODE$  is greater than the acute angle  $ODC$ : therefore the angle  $ODE$  is greater than the angle  $OED$ . Therefore  $OD$  is less than  $OE$ , by I. 19.

Lastly, from  $C$  on the straight line  $AB$  take  $CF$  equal to  $CD$ , and on the other side of  $C$ ; join  $OF$ . Then  $OF$  is equal to  $OD$  by I. 4. And no other straight line can be drawn from  $O$  to  $AB$  equal to  $OD$ , besides  $OF$ .

For if this straight line were nearer to  $OC$  than  $OD$  or  $OF$  is it would be less than  $OD$ ; and if it were more remote from  $OC$  than  $OD$  or  $OF$  is, it would be greater than  $OD$ .

22. Let  $ABC$  be a triangle, and  $O$  any point. Join  $OA$ ,  $OB$ ,  $OC$ . Then  $OA$  and  $OB$  are together greater than  $AB$ ,  $OB$  and  $OC$  are together greater than  $BC$ , and  $OC$  and  $OA$  are together greater than  $CA$ , by I. 20. Hence twice the sum of  $OA$ ,  $OB$ , and  $OC$  is greater than the sum of  $AB$ ,  $BC$ , and  $CA$ ; therefore the sum of  $OA$ ,  $OB$ , and  $OC$  is greater than half the sum of  $AB$ ,  $BC$ , and  $CA$ .

23. Let  $ABCD$  be a quadrilateral figure. Draw the diagonals  $AC$  and  $BD$ . Then  $AB$  and  $BC$  are greater than  $AC$ , and  $AD$  and  $DC$  are greater than  $AC$ ; therefore the four sides  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  are greater than twice  $AC$ . Similarly it may be shewn that the four sides are greater than twice  $BD$ . Hence twice the sum of the four sides is greater than twice the sum of the diagonals; therefore the sum of the four sides is greater than the sum of the diagonals.

24. Let  $ABC$  be a triangle, and  $D$  the middle point of the base  $BC$ ; join  $AD$ : then  $AB$  and  $AC$  together shall be greater than twice  $AD$ .

Produce  $AD$  to a point  $E$  so that  $DE$  may be equal to  $AD$ ; join  $BE$ . In the two triangles  $ADC$  and  $EDB$  the two sides  $AD$ ,  $DC$  are equal to the two sides  $ED$ ,  $DB$  each to each; and the angle  $ADC$  is equal to the angle  $EDB$ , by I. 15: therefore  $AC$  is equal to  $BE$ , by I. 4. The two sides  $AB$ ,  $BE$  are greater than  $AE$ , by I. 20; therefore the two sides  $AB$ ,  $AC$  are greater than  $AE$ , that is greater than twice  $AD$ .

25. Let  $ABC$  be a triangle in which the angle  $C$  is equal to the sum of the angles  $A$  and  $B$ . At the point  $C$  in the straight line  $AC$  make the angle  $ACG$  equal to the angle  $A$ , and let  $CG$  meet  $AB$  at  $D$ . Then the triangle  $ACD$  is isosceles, by I. 6. Also as the angle  $ACB$  is equal to the sum of the angles  $A$  and  $B$ , and the angle  $ACG$  is equal to the angle  $A$ , the angle  $BCG$  is equal to the angle  $B$ : therefore the triangle  $BCD$  is isosceles, by I. 6.

26. It is shewn in the preceding Exercise that  $ACD$  is an isosceles triangle, having  $AD$  equal  $CD$ ; also that  $BCD$  is an isosceles triangle having  $BD$  equal to  $CD$ . Hence, as  $AD$  and  $BD$  are each equal to  $CD$ , the point  $D$  is the middle point of  $AB$ , and  $AB$  is equal to twice  $CD$ .

27. Let  $AB$  be the given base; at the point  $A$  make the angle  $DAB$  equal to the given angle. From  $AD$  cut off  $AE$  equal to the given sum of the sides; join  $EB$ . At the point  $B$  make the angle  $EBF$  equal to the angle  $AEB$ , and on the same side of  $EB$ ; let  $BF$  meet  $AE$  at  $C$ : then  $ACB$  will be the triangle required.

For, since the angle  $EBC$  is equal to the angle  $BEC$ , the sides  $EC$  and  $BC$  are equal, by I. 6. Therefore the sum of  $AC$  and  $CB$  is equal to the sum

of  $AC$  and  $CE$ , that is to the given sum of the sides. Also the base  $AB$  and the angle  $BAC$  have the required values.

28. From any point  $D$  in the straight line bisecting the angle  $A$  of a triangle draw  $DE$  perpendicular to the side  $AB$ , and  $DF$  perpendicular to the side  $AC$ . Then in the triangles  $DAE$  and  $DAF$ , the side  $DA$  is common; the angle  $DAE$  is equal to the angle  $DAF$ , by hypothesis; and the right angles  $DEA$  and  $DFA$  are equal: therefore  $DE$  is equal to  $DF$  by I. 26.

29. Let  $AB$  be the given straight line in which the point is to be found. Let  $CD$  and  $EF$  be the other two straight lines, and let them meet, produced if necessary, at  $O$ . Through  $O$  draw a straight line bisecting the angle between  $CD$  and  $EF$ ; and let  $AB$ , produced if necessary, meet this straight line at  $K$ ; then  $K$  will be such a point as is required: for the perpendiculars from  $K$  on the straight lines  $CD$  and  $EF$  may be shewn to be equal in the manner of the preceding Exercise.

Two straight lines can be drawn through  $O$  bisecting angles formed by the given straight lines, so that in general two solutions of the problem can be obtained, but there will be only one solution if  $AB$  is parallel to either of the bisecting straight lines.

If  $CD$  and  $EF$  are parallel the construction fails. We must then draw a straight line  $KL$  parallel to  $CD$  and  $EF$ , and midway between them: the intersection of this straight line with  $AB$ , produced if necessary, will be the required point. But if  $KL$  is parallel to  $AB$  there will be no solution.

30. Suppose  $A$  the given point through which the straight line is to be drawn, and  $B$  and  $C$  the other given points from which perpendiculars are to be drawn. Join  $BC$ , and bisect it at  $D$ ; join  $AD$ : this shall be the required straight line.

For draw  $BE$  and  $CF$  perpendicular to  $AD$ , produced if necessary. Then in the triangles  $BDE$ ,  $CDF$  the sides  $BD$  and  $CD$  are equal; the angles  $BDE$  and  $CDF$  are equal, by I. 15; and the right angles  $BED$  and  $CFD$  are equal: therefore  $BE$  is equal to  $CF$ , by I. 26.

31. In the triangles  $ADB$ ,  $ADE$  the side  $AD$  is common; the angles  $BAD$  and  $EAD$  are equal by hypothesis; and the right angles  $ADB$  and  $ADE$  are equal: therefore  $BD$  is equal to  $ED$ , by I. 26.

32. Bisect the angle  $BAC$  by the straight line  $AD$ ; from  $P$  draw  $PG$  perpendicular to  $AD$ , and produce  $PG$  both ways to meet  $AB$  at  $E$ , and  $AC$  at  $F$ : then  $AE$  will be equal to  $AF$ .

For in the two triangles  $AGE$ ,  $AGF$  the side  $AG$  is common; the angles  $EAG$  and  $FAG$  are equal, by construction; and the right angles  $EGA$  and  $FGA$  are equal: therefore  $AE$  is equal to  $AF$ , by I. 26.

33. Let  $ABC$  be a triangle having the angle  $B$  a right angle, and let  $DEF$  be a triangle having the angle  $E$  a right angle; also let  $AC$  be equal to  $DF$ , and  $AB$  equal to  $DE$ : then shall the triangles  $ABC$  and  $DEF$  be equal in all respects.

Produce  $CB$  to  $G$ , so that  $BG$  may be equal to  $EF$ ; and join  $AG$ . Then the angle  $ABG$  is a right angle, by I. 13, and is therefore equal to the angle  $DEF$ ; also the sides  $AB$ ,  $BG$  are equal to the sides  $DE$ ,  $EF$  each to each: therefore the triangles  $ABG$ ,  $DEF$  are equal in all respects, so that  $AG$  is

equal to  $DF$ . But  $AC$  is equal to  $DF$  by hypothesis; therefore  $AC$  is equal to  $AG$ , and the angle  $ACG$  is equal to the angle  $AGC$ . Therefore the two triangles  $ABC$ ,  $ABG$  are equal in all respects, by I. 26. But the triangles  $ABG$ ,  $DEF$  were shewn to be equal in all respects; therefore the triangles  $ABC$ ,  $DEF$  are equal in all respects.

## I. 27 to 31.

34. Let  $ABC$  be a triangle having the sides  $AB$  and  $AC$  equal. Draw any straight line parallel to  $BC$ , meeting  $AB$  at  $D$ , and  $AC$  at  $E$ ; then the angle  $ADE$  shall be equal to the angle  $AED$ .

For the angle  $ADE$  is equal to the angle  $ABC$ , and the angle  $AED$  is equal to the angle  $ACB$ , by I. 29. But the angle  $ABC$  is equal to the angle  $ACB$ , by I. 5. Therefore the angle  $ADE$  is equal to the angle  $AED$ .

35. Let the straight lines  $A$  and  $B$  meet at  $K$ , let the straight lines  $C$  and  $D$  meet at  $L$ , and let the straight lines  $A$  and  $D$  meet at  $M$ . The acute angle at  $K$  is equal to the acute angle at  $M$ , by I. 29; and the acute angle at  $M$  is equal to the acute angle at  $L$  also, by I. 29. Therefore the acute angle at  $K$  is equal to the acute angle at  $L$ .

36. Let the straight line  $AB$  be terminated by two parallel straight lines. Let  $C$  be the middle point of  $AB$ ; through  $C$  draw any straight line  $DCE$ , terminated by the same parallel straight lines as  $AB$ , so that  $AD$  is parallel to  $EB$ . Then will  $ED$  be bisected at  $C$ .

For in the two triangles  $ACD$ ,  $BCE$  the two sides  $AC$ ,  $BC$  are equal by hypothesis; the angles  $ACD$ ,  $BCE$  are equal by I. 15; and the angles  $CAD$ ,  $CBE$  are equal by I. 29: therefore the triangles are equal in all respects by I. 26, so that  $CD$  is equal to  $CE$ .

37. Let  $O$  be a point equidistant from two parallel straight lines; through  $O$  draw one straight line  $AOB$  terminated by the parallels, and also another straight line  $COD$  terminated by the parallels, so that  $A$  and  $C$  are on one of the parallels, and  $B$  and  $D$  on the other: then will  $AC$  be equal to  $BD$ .

Since the given straight lines are parallel, a straight line can be drawn through  $O$  to meet the parallels at right angles, and this straight line will be bisected at  $O$  because  $O$  is equidistant from the parallels, by hypothesis. Therefore by Exercise 36 the straight lines  $AB$  and  $DC$  are bisected at  $O$ . Thus in the two triangles  $AOC$ ,  $BOD$  the two sides  $AO$ ,  $OC$  are equal to the two sides  $BO$ ,  $OD$  each to each; and the angle  $AOC$  is equal to the angle  $BOD$ , by I. 15: therefore  $AC$  is equal to  $BD$ , by I. 4.

38. Let  $ABC$  be a triangle: produce  $BA$  to  $D$ ; suppose that  $AE$  bisects the angle  $DAC$ , and that it is parallel to  $BC$ : then will  $ABC$  be an isosceles triangle.

For since  $AE$  is parallel to  $BC$  the angle  $DAE$  is equal to the angle  $ABC$ , by I. 29, and also the angle  $CAE$  is equal to the angle  $ACB$ , by I. 29. But the angle  $DAE$  is equal to the angle  $EAC$ , by hypothesis; therefore the angle  $ABC$  is equal to the angle  $ACB$ : therefore the side  $AB$  is equal to the side  $AC$ , by I. 6.

39. Take any point  $E$  in  $DC$ , and at the point  $E$  make the angle  $CEF$  equal to the given angle. Through  $A$  draw a straight line parallel to  $FE$ , and meeting  $CD$  at  $B$ : then  $B$  is the required point.

For the angle  $ABC$  is equal to the angle  $FEC$  by I. 29; and therefore the angle  $ABC$  is equal to the given angle.

40. Let  $ABC$  be a triangle; let a straight line be drawn bisecting the angle  $A$ , and meeting  $BC$  at  $D$ . From  $D$  draw a straight line parallel to  $AB$ , meeting  $AC$  at  $F$ , and also a straight line parallel to  $AC$ , meeting  $AB$  at  $E$ : then  $DE$  shall be equal to  $DF$ .

For in the triangles  $AED$ ,  $AFD$  the side  $AD$  is common; the angle  $EAD$  is equal to the angle  $FAD$ , by construction; the angle  $EDA$  is equal to the angle  $DAF$ , and the angle  $FDA$  to the angle  $DAE$ , by I. 29; so that the angle  $EDA$  is equal to the angle  $FDA$ ; hence the two triangles are equal in all respects by I. 26. Thus  $DE$  is equal to  $DF$ .

41. The angle  $FEC$  is equal to the angle  $ECB$ , by I. 29; the angle  $ECB$  is equal to the angle  $ECF$ , by hypothesis; therefore the angle  $FEC$  is equal to the angle  $ECF$ . Therefore  $EF$  is equal to  $FC$ , by I. 6. Again, the angle  $FGC$  is equal to the angle  $GCD$ , by I. 29; the angle  $GCD$  is equal to the angle  $FCG$ , by hypothesis; therefore the angle  $FGC$  is equal to the angle  $FCG$ . Therefore  $FG$  is equal  $FC$ , by I. 6. And it has been shewn that  $FE$  is equal to  $FC$ ; therefore  $EF$  is equal to  $FG$ .

42. Bisect the angle  $ABC$  by a straight line meeting  $AC$  at  $E$ ; through  $E$  draw a straight line parallel to  $CB$  meeting  $AB$  at  $D$ : then  $D$  shall be the point required.

For the angle  $DEB$  is equal to the angle  $EBC$ , by I. 29; the angle  $DBE$  is equal to the angle  $EBC$ , by construction; therefore the angle  $DEB$  is equal to the angle  $DBE$ ; therefore  $DB$  is equal to  $DE$ , by I. 6. And the angle  $DEA$  is equal to the angle  $BCA$  by I. 29, and is therefore a right angle; so that  $DE$  is perpendicular to  $AC$ .

43. Bisect the angle  $ABC$  by a straight line, meeting  $AC$  at  $E$ ; through  $E$  draw a straight line parallel to  $BC$ , meeting  $AB$  at  $D$ . Then will  $BD$ ,  $DE$ ,  $EC$  be all equal.

For the angle  $DEB$  is equal to the angle  $EBC$ , by I. 29; the angle  $EBC$  is equal to the angle  $DBE$  by construction; therefore the angle  $DEB$  is equal to the angle  $DBE$ . Therefore  $DB$  is equal to  $DE$ , by I. 6. Again, the angle  $ADE$  is equal to the angle  $ABC$ , and the angle  $AED$  is equal to the angle  $ACB$ , by I. 29; also the angle  $ABC$  is equal to the angle  $ACB$ : therefore the angle  $ADE$  is equal to the angle  $AED$ . Therefore  $AD$  is equal to  $AE$ , by I. 6. But the whole  $AB$  is equal to the whole  $AC$ ; therefore  $DB$  is equal to  $EC$ . Thus  $BD$ ,  $DE$ ,  $EC$  are all equal.

44. From  $A$  draw a straight line bisecting the angle  $BAC$ , and meeting  $BC$  at  $F$ . Then the triangles  $BAF$  and  $CAF$  are equal in all respects, by I. 4; so that the angles  $AFB$ ,  $AFC$  are equal, and therefore each of them is a right angle. Therefore  $AF$  is parallel to  $ED$ , by I. 28. The angle  $AED$  is equal to the angle  $CAF$ , and the angle  $EDA$  is equal to the angle  $BAF$ , by I. 29. But the angle  $BAF$  is equal to the angle  $CAF$ , by construction; therefore the angle  $AED$  is equal to the angle  $ADE$ : therefore  $AE$  is equal to  $AD$ , by I. 6.

## I. 32.

45. Let  $ABC$  be a triangle having the sides  $AB$  and  $AC$  equal. From  $B$  draw a perpendicular to  $AC$  meeting  $AC$  at  $D$ , and from  $C$  draw a perpendicular to  $AB$  meeting it at  $E$ . Then each of the angles  $CBD$ ,  $BCE$  will be equal to half the angle  $A$ .

Bisect the angle  $A$  by a straight line meeting the base  $BC$  at  $F$ . Then the triangles  $BAF$ ,  $CAF$  are equal in all respects, by I. 4; and therefore the angle  $AFB$  is a right angle. Then in the triangles  $BAF$ ,  $BCE$  the angle  $ABC$  is common; the right angle  $AFB$  is equal to the right angle  $CEB$ : therefore the third angle  $BAF$  is equal to the third angle  $BCE$ , by I. 32. Similarly the angle  $CAF$  is equal to the angle  $CBD$ .

46.  $AC$  is equal to  $CE$ , and  $BC$  is equal to  $CD$ , by construction. The angle  $ACE$  is equal to the angle  $BCD$ , each being one-third of two right angles, by I. 32; to each of these add the angle  $ACB$ : therefore the whole angle  $ACD$  is equal to the whole angle  $BCE$ . Thus in the two triangles  $ACD$ ,  $ECB$  the two sides  $AC$ ,  $CD$  are equal to the two sides  $EC$ ,  $CB$  each to each; and the included angles are equal: therefore  $AD$  is equal to  $BE$ . Similarly  $AD$  is equal to  $FC$ .

47. The figure has eight equal sides, and eight equal angles: the interior angles of the figure together with four right angles are equal to sixteen right angles: therefore all the interior angles of the figure are equal to twelve right angles, by I. 32, Corollary I. Hence each angle is twelve-eighths of a right angle, that is, a right angle and a half.

48. Let  $A$  and  $B$  be the two given points,  $CD$  the given straight line. At the point  $C$  make the angle  $ECD$  equal to the angle of an equilateral triangle; and at the point  $D$  make the angle  $FDC$  also equal to the angle of an equilateral triangle, and on the same side of  $CD$  as the angle  $ECD$ . Through  $A$  draw a straight line parallel to  $EC$ , meeting  $CD$ , produced if necessary, at  $G$ : through  $B$  draw a straight line parallel to  $FD$  meeting  $CD$ , produced if necessary, at  $H$ . Produce  $GA$  and  $HB$  to meet at  $K$ : then  $GKH$  is the equilateral triangle required.

For since the angle at  $G$  is a third of two right angles, and so also is the angle at  $H$ , the angle at  $K$  is also a third of two right angles, by I. 32. Hence the triangle  $GKH$  is equiangular, and therefore also equilateral by I. 6.

49. Let  $ABC$  be a triangle, having  $AB$  equal to  $AC$ . Bisect the angles  $B$  and  $C$  by straight lines meeting at  $D$ . Produce  $CB$  to any point  $E$ . The angle  $DBE$  is equal to the two angles  $BDC$ ,  $DCB$  by I. 32; but  $DCB$  is half the angle  $ACB$ , and is therefore equal to half the angle  $ABC$ , and is therefore equal to the angle  $ABD$ . Therefore the angle  $DBE$  is equal to the two angles  $ABD$  and  $BDC$ . Take away the common angle  $ABD$ ; therefore the angle  $ABE$  is equal to the angle  $BDC$ .

50. The angle  $ACB$  is equal to the angle  $ABC$ , and the angle  $ACD$  is equal to the angle  $ADC$ , by I. 5. Therefore the angles  $ACB$  and  $ACD$  together are equal to the angles  $ABC$  and  $ADC$  together. But the angles  $ACB$ ,  $ACD$ ,  $ABC$ ,  $ADC$  are together equal to two right angles, by I. 32. Therefore the angles  $ACB$  and  $ACD$  are together equal to one right angle.

51. Produce  $AB$  to any point  $H$ , and  $AC$  to any point  $K$ ; bisect the angle  $CBH$  by the straight line  $BD$ , and bisect the angle  $BCK$  by the straight line  $CD$ : then the angle  $BDC$ , together with half the angle  $BAC$ , will make up a right angle.

Bisect the angle  $ABC$  by the straight line  $BE$ , and bisect the angle  $ACB$  by the straight line  $CE$ ; then the angles  $EBD$  and  $ECD$  will be right angles. For the angle  $EBC$  is half the angle  $ABC$ , and the angle  $ECB$  is half the angle  $ACB$ ; therefore the angles  $EBC$  and  $ECB$  are together half the angles  $ABC$  and  $ACB$ , that is equal to a right angle, by I. 13: thus  $EBD$  is a right angle. Similarly  $ECD$  is a right angle. The angles  $BEC$ ,  $ERC$ ,  $ECB$  are together equal to two right angles, by I. 32; that is  $BEC$  together with half  $ABC$  and half  $ACB$  are equal to two right angles. Join  $ED$ . Then  $BED$  and  $EDB$  are together equal to a right angle, by I. 32;  $CED$  and  $EDC$  are together equal to a right angle, by I. 32: therefore  $BEC$  and  $BDC$  are together equal to two right angles. Thus  $BEC$  and  $BDC$  are equal to  $BEC$  together with half  $ABC$  and half  $ACB$ . Therefore  $BDC$  is equal to half  $ABC$  and half  $ACB$ . Therefore  $BDC$  together with half  $BAC$  is equal to half  $BAC$ , half  $ABC$ , and half  $ACB$ , that is equal to half two right angles, that is equal to a right angle.

52. Let  $ABC$  be a triangle. Suppose the angle  $ABC$  greater than the sum of the other two angles; then twice the angle  $ABC$  is greater than the sum of the angles  $ABC$ ,  $BCA$ ,  $CAB$ , that is greater than two right angles; therefore the angle  $ABC$  is greater than a right angle. Again, suppose the angle  $ABC$  equal to the sum of the other two angles; then twice the angle  $ABC$  is equal to the sum of the angles  $ABC$ ,  $BCA$ ,  $CAB$ , that is equal to two right angles: therefore the angle  $ABC$  is a right angle. Lastly, suppose the angle  $ABC$  less than the sum of the other two angles; then twice the angle  $ABC$  is less than the sum of the angles  $ABC$ ,  $BCA$ ,  $CAB$ , that is less than two right angles: therefore the angle  $ABC$  is less than a right angle.

53. Construct an equilateral triangle  $ABC$ . Bisect the angle  $A$  by a straight line  $AD$ , and bisect the angle  $C$  by a straight line  $CD$ . Then  $ADC$  will be such a triangle as is required.

For the angle  $DAC$ , being half the angle of an equilateral triangle, is one sixth of two right angles; so also is the angle  $DCA$ ; therefore the angle  $ADC$  is four sixths of two right angles, by I. 32. Thus the angle  $ADC$  is four times each of the angles  $DAC$ ,  $DCA$ .

54. Since  $BC$  is bisected at  $E$  the two sides  $AE$ ,  $EC$  are equal to the two sides  $FE$ ,  $EB$  each to each; the angle  $AEC$  is equal to the angle  $FEB$ , by I. 15; therefore the triangles  $AEC$  and  $FEB$  are equal in all respects, so that the angle  $ACE$  is equal to the angle  $FBE$ .

In a similar way by comparing the triangles  $CGA$  and  $HGB$ , we see that the angle  $CAG$  is equal to the angle  $HGB$ .

Therefore the angles  $FBE$ ,  $EBG$ ,  $GBH$  are together equal to the angles of the triangle  $ABC$ , that is to two right angles, by I. 32. Therefore the angles  $ABF$  and  $ABH$  are together equal to two right angles: therefore  $HB$  and  $BF$  are in the same straight line, by I. 14.

55. Take any straight line  $AB$ . At the point  $A$  make the angle  $BAD$  equal to a right angle. Bisect the angle  $BAD$  by the straight line  $AE$ ; and

bisect the angle  $DAE$  by the straight line  $AF$ ; I. 9. Then the angle  $BAF$  is three fourths of a right angle. At the point  $B$  make the angle  $ABG$  equal to the angle  $BAF$ , and on the same side of  $AB$ ; let  $AF$  and  $BG$  meet at  $C$ : then  $ABC$  will be the triangle required.

For the two angles  $ABC$  and  $BAC$  are by construction together equal to three halves of a right angle, therefore the angle  $ACB$  is half a right angle by I. 32. Thus the half of the angle  $ACB$  is a fourth of a right angle, and is therefore equal to one third of each of the angles  $BAC$  and  $ABC$ .

56. On  $AB$  measure off  $AD$  equal to the given straight line. At the point  $D$  draw  $DQ$  making with  $AD$  the angle  $ADQ$  equal to half the given angle, and meeting  $AC$  at  $Q$ . At  $Q$  draw  $QP$ , on the same side of  $QD$  as  $DA$  is, making the angle  $DQP$  equal to the angle  $QDP$ , and meeting  $AB$  at  $P$ . Then  $PQ$  is equal to  $PD$ , by I. 6; so that  $AP$  and  $PQ$  together are equal to  $AD$ , that is to the given straight line. And the angle  $APQ$  is equal to the sum of the angles  $PDQ$  and  $PQD$ , and is therefore equal to the given angle.

57. Let  $ABC$  be a triangle having the sides  $AB$  and  $AC$  equal. From  $B$  draw a straight line making the angle  $DBC$  equal to one third of the angle  $ABC$ , on the other side of  $BC$ , and meeting  $AC$  produced at  $D$ . From  $C$  draw a straight line making the angle  $ECB$  equal to one third of the angle  $ACB$ , on the other side of  $CB$ , and meeting  $AB$  produced at  $E$ . Let  $BD$  and  $CE$  intersect at  $F$ .

The triangle  $BFC$  has obviously the angles  $BCF$  and  $CBF$  equal, and is therefore isosceles by I. 6.

The angle  $BFE$  is equal to the sum of the angles  $BCF$  and  $CBF$ , by I. 32, and is therefore equal to two thirds of the angle  $ABC$ . The angle  $BEC$  is equal to the difference of the angles  $ABC$  and  $BCE$ , by I. 32, and is therefore equal to two thirds of the angle  $ABC$ . Therefore the angles  $BFE$  and  $BEF$  are equal, and the triangle  $BFE$  is isosceles, by I. 6. Similarly the triangle  $CFD$  is isosceles.

58. The angle  $AEC$  is equal to the sum of the angles  $ECB$  and  $EBC$  by I. 32, and so also is the angle  $DEB$ : therefore the angles  $ECB$  and  $EBC$  are together half the angles  $AEC$  and  $DEB$ . The angles  $ECF$  and  $EBF$  are together half the angles  $ECA$  and  $EBD$ , by construction. Hence the angles  $ECB$ ,  $EBC$ ,  $ECF$ ,  $EBF$  are together half the angles  $AEC$ ,  $DEB$ ,  $ECA$ ,  $EBD$ . Take the former sum from two right angles, and the remainder is the angle  $BFC$ ; take the latter sum from two right angles, and the remainder is half  $EAC$  and  $EDB$ ; therefore the angle  $BFC$  is half the sum of the angles  $EAC$ ,  $EDB$ .

59. Let  $ABC$  be a triangle having the angle  $ACB$  a right angle. At the point  $C$  draw a straight line  $CD$  making the angle  $ACD$  equal to the angle  $CAB$ , and meeting  $AB$  at  $D$ . Since the angle  $ACB$  is a right angle it is equal to the sum of the two angles  $CAB$  and  $CBA$ , by I. 32; the angle  $ACD$  is equal to the angle  $CAB$ , by construction: therefore the remaining angle  $BCD$  is equal to the angle  $CBA$ . Because the angles  $CAD$  and  $ACD$  are equal, the sides  $AD$  and  $CD$  are equal, by I. 6; and because the angles  $CBD$  and  $BCD$  are equal, the sides  $BD$  and  $CD$  are equal, by I. 6. Hence  $AD$ ,  $BD$ , and  $CD$  are all equal; so that  $D$  is the middle point of  $AB$ , and  $CD$  is equal to half of  $AB$ .

60. Let  $F$  be the middle point of  $AB$ : then will  $EF$  be equal to  $DF$ .  
For  $EF$  and  $DF$  are each equal to half of  $AB$ , by Exercise 59; therefore  $EF$  is equal to  $DF$ .

61. Use the diagram drawn for Exercise 60; from  $F$  draw  $FG$  perpendicular to  $ED$ ; then will  $EG$  be equal to  $DG$ . For  $FE$  is equal to  $FD$ , by Exercise 60; therefore the angle  $FEG$  is equal to the angle  $FDG$ , by I. 5; and the right angle  $FGE$  is equal to the right angle  $FGD$ : therefore the triangle  $FEG$  is equal to the triangle  $FDG$  in all respects, by I. 26. Thus  $EG$  is equal to  $DG$ .

62. In the diagram of I. 1, let  $BA$  be produced through  $A$  to meet the circle at  $K$ . Join  $CK$ ,  $HK$ ,  $AH$ ,  $CH$ .

The angle  $CAB$  is one third of two right angles; therefore the angle  $CAK$  is two thirds of two right angles. Also, each of the angles  $CAB$ ,  $HAB$  being one third of two right angles, the whole angle  $CAH$  is two thirds of two right angles. Therefore the angle  $CAK$  is equal to the angle  $CAH$ .

In the two triangles  $CAK$ ,  $CAH$  the two sides  $CA$ ,  $AK$  are equal to the two sides  $CA$ ,  $AH$  each to each; and the angle  $CAK$  is equal to the angle  $CAH$ : therefore  $CK$  is equal to  $CH$ , by I. 4.

Similarly it may be shewn that  $HK$  is equal to  $HC$ . Thus  $KC$ ,  $CH$ , and  $HK$  are all equal.

63. Let  $AB$  and  $AC$  be the equal sides; let  $BD$  bisect the angle  $ABC$ ; and let  $CE$  bisect the angle  $ACB$ . Join  $DE$ .

In the triangles  $BCD$ ,  $CBE$  the angle  $BCD$  is equal to the angle  $CBE$ , and the angle  $DBC$  is equal to the angle  $ECB$ ; therefore these triangles are equal in all respects, by I. 26, so that  $CD$  is equal to  $BE$ . But  $AC$  is equal to  $AB$ ; therefore  $AD$  is equal to  $AE$ ; therefore the angle  $AED$  is equal to the angle  $ADE$ . Also the angle  $ABC$  is equal to the angle  $ACB$ . Therefore the angle  $AED$  is equal to the angle  $ABC$ , by I. 32. Therefore  $ED$  is parallel to  $BC$ , by I. 28.

64. In  $AC$  take a point  $D$  such that  $AD$  is equal to  $AP$ ; join  $DP$ ; in  $AD$  produced, take a point  $Q$  such that  $DQ$  is equal to  $DP$ ; join  $PQ$ : then will the angle  $APQ$  be equal to three times the angle  $AQP$ .

Since  $DP$  is equal to  $DQ$  the angle  $DPQ$  is equal to the angle  $DQP$ , by I. 5. The angle  $ADP$  is equal to the sum of the angles  $DPQ$  and  $DQP$ , by I. 32: therefore the angle  $ADP$  is twice the angle  $DQP$ . Therefore the angle  $APD$  is twice the angle  $DQP$ , by I. 5. To the former add the angle  $DQP$ , which is equal to the angle  $DQP$ : therefore the angle  $APQ$  is three times the angle  $AQP$ .

65. Take a straight line  $AD$  equal to the given sum of the sides. At the point  $D$  draw a straight line  $DE$  making the angle  $ADE$  equal to half a right angle. With centre  $A$ , and radius equal to the given hypotenuse, describe a circle cutting  $DE$  at a point  $B$ . From  $B$  draw  $BC$  perpendicular to  $AD$ . Then  $ACB$  will be such a triangle as is required.

For  $ACB$  is a right angle by construction; and the hypotenuse  $AB$  is of the required length. Also since the angle  $BCD$  is a right angle, and the angle  $BDC$  is half a right angle, the angle  $CBD$  is half a right angle, and is therefore equal to the angle  $BDC$ : therefore  $BC$  is equal to  $CD$ , by I. 6. Thus the sum of the sides  $AC$  and  $CB$  is equal to  $AD$ , that is to the given sum.

In order that the construction may be possible the given hypotenuse must



not be less than the perpendicular from  $A$  on  $DE$ ; and if this condition is satisfied there will be two intersections of the circle with  $DE$ , and thus *two solutions apparently*: but it will be found on examination that there is only one distinct solution.

66. Take a straight line  $AD$  equal to the given difference of the sides. At the point  $D$  draw a straight line  $DE$  making with  $AD$ , produced through  $D$ , an angle equal to half a right angle. With centre  $A$ , and radius equal to the given hypotenuse, describe a circle cutting  $DE$  at  $B$ . From  $B$  draw  $BC$  perpendicular to  $AD$  produced. Then  $ACB$  will be such a triangle as is required.

For  $ACB$  is a right angle by construction; and the hypotenuse  $AB$  is of the required length. Also since the angle  $BCD$  is a right angle, and the angle  $BDC$  is half a right angle, the angle  $DBC$  is half a right angle, and is therefore equal to the angle  $BDC$ : therefore  $BC$  is equal to  $CD$ . Thus the difference of the sides is equal to  $AD$ , that is to the given difference.

In order that the construction may be possible the given hypotenuse must be greater than the given difference of the sides.

67. Let  $AB$  be the given hypotenuse. Bisect  $AB$  at  $D$ ; from  $D$  draw a straight line at right angles to  $AB$ , and on it take  $DE$  equal to the given perpendicular. Through  $E$  draw  $FEG$  parallel to  $AB$ . From centre  $D$ , with radius equal to  $AD$ , describe a circle cutting  $FG$  at  $C$ . Join  $CA$  and  $CB$ : then  $ACB$  shall be the triangle required.

For the angle  $ACD$  is equal to the angle  $CAD$ , and the angle  $BCD$  is equal to the angle  $CBD$ , by I. 5; therefore the whole angle  $ACB$  is equal to the sum of the angles  $CAB$  and  $CBA$ : therefore the angle  $ACB$  is a right angle, by I. 32. Thus  $ACB$  is a right-angled triangle having the given hypotenuse.

From  $C$  draw  $CH$  perpendicular to  $AB$ ; then will  $CH$  be equal to  $ED$ . The angles  $EDH$  and  $CHD$  are together equal to two right angles; therefore  $CH$  is parallel to  $ED$ , by I. 28; therefore the angle  $EDC$  is equal to the angle  $HCD$ , by I. 29. Therefore the two triangles  $EDC$  and  $HCD$  are equal in all respects, by I. 26; so that  $CH$  is equal to  $ED$ . Thus the perpendicular from the right angle on the hypotenuse has the given length.

68. Let  $DE$  be the given perimeter, and  $FGH$  the given angle. Draw  $GK$  at right angles to  $GH$ , and on the same side of it as  $GF$ . At the point  $D$  draw  $DL$ , making the angle  $LDE$  equal to half the angle  $FGH$ ; at the point  $E$  draw  $EM$  on the same side of  $DE$  as  $DL$ , making the angle  $MED$  equal to half the angle  $FGK$ . Let  $DL$  and  $ME$  meet at  $C$ . Draw  $CA$ , meeting  $DE$  at  $A$ , and making the angle  $DCA$  equal to the angle  $CDA$ ; draw  $CB$ , meeting  $ED$  at  $B$ , and making the angle  $ECB$  equal to the angle  $CEB$ : then  $ABC$  will be the required triangle.

For the angle  $CAB$  is equal to the sum of the angles  $ACD$ ,  $ADC$ ; that is to twice the angle  $ADC$ ; that is to the angle  $FGH$ . Similarly the angle  $CBA$  is equal to the angle  $FGK$ . Thus the two angles  $CAB$  and  $CBA$  are together equal to a right angle; and therefore the angle  $ACB$  is a right angle by I. 32.

The side  $AC$  is equal to  $AD$ , and the side  $BC$  to  $BE$ , by I. 6. Therefore the sum of the sides  $AC$ ,  $BC$ , and  $AB$  is equal to  $DE$ , that is to the given perimeter.

69. Let  $BAC$  be a right angle. On  $AB$  describe an equilateral triangle  $ADB$ ; bisect the angle  $BAD$  by the straight line  $AE$ ; then will the angles  $BAE$ ,  $EAD$ ,  $DAC$  be all equal.

For the angle  $BAC$  is a right angle; the angle  $BAD$  is one third of two right angles, that is two thirds of one right angle, by I. 32; therefore the angle  $CAD$  is one third of a right angle. And as the angle  $BAD$  is bisected by  $AE$  the angle  $BAE$  is equal to the angle  $EAD$ , each being one third of a right angle. Hence the angles  $BAE$ ,  $EAD$ ,  $DAC$  are all equal, each being one third of a right angle.

70. Let  $AB$  be the given straight line. On  $AB$  describe an equilateral triangle  $ABC$ . Bisect the angle  $CAB$  by the straight line  $AD$ , and bisect the angle  $CBA$  by the straight line  $BD$ . Through  $D$  draw a straight line parallel to  $CA$ , meeting  $AB$  at  $E$ ; and through  $D$  draw a straight line  $DF$  parallel to  $CB$ , meeting  $AB$  at  $F$ . Then will  $AE$ ,  $EF$ ,  $FB$  all be equal.

Since  $DE$  is parallel to  $CA$  the angle  $EDA$  is equal to the angle  $DAC$ , by I. 29; but the angle  $DAE$  is equal to the angle  $DAC$ , by construction; therefore the angles  $EDA$  and  $DAE$  are equal: therefore  $AE$  is equal to  $ED$ , by I. 6. Similarly it may be shewn that  $BF$  is equal to  $FD$ .

Because  $DE$  is parallel to  $CA$  the angle  $DEF$  is equal to the angle  $CAB$ , by I. 29. Similarly the angle  $DFE$  is equal to the angle  $CBA$ . Therefore the angle  $EDF$  is equal to the angle  $ACB$ , by I. 32. Thus the triangle  $EDF$  is equiangular; and therefore it is equilateral by I. 6.

Now  $AE$  was shewn to be equal to  $ED$ ; therefore  $AE$  is equal to  $EF$ . Similarly  $BF$  is equal to  $FE$ . Thus  $AE$ ,  $EF$ ,  $FB$  are all equal.

71. Let  $LM$  and  $PQ$  be the parallel straight lines; and  $A$  the given point. Suppose  $A$  to be between the parallel straight lines. Through  $A$  draw a straight line perpendicular to one of the parallel straight lines, and therefore also perpendicular to the other by I. 29. Let this straight line meet  $LM$  at  $B$ , and  $PQ$  at  $C$ . From  $B$  on  $LM$  take  $BD$  equal to  $AC$ ; and from  $C$  on  $PQ$  take  $CE$  equal to  $AB$ , and on the same side of  $BC$  as  $BD$  is. Join  $AD$  and  $AE$ : these will be the required straight lines.

For the triangles  $BAD$ ,  $CEA$  are equal in all respects by I. 4; so that  $AD$  is equal to  $AE$ , and the angle  $CAE$  is equal to the angle  $BDA$ . But the angles  $BAD$  and  $BDA$  are together equal to a right angle, by I. 32: therefore the angles  $BAD$  and  $CAE$  are together equal to a right angle. Therefore the angle  $EAD$  is a right angle, by I. 13.

If  $A$  is not between the parallel straight lines,  $CE$  and  $BD$  must be taken on opposite sides of  $BC$ .

72. Let  $ABC$  be the given triangle, and  $DE$  the given perimeter. At the point  $D$  make the angle  $LDE$  equal to half the angle  $ABC$ ; and at the point  $E$ , on the same side of  $ED$ , make the angle  $MED$  equal to half the angle  $ACB$ . Let  $DL$  and  $EM$  meet at  $F$ . From  $F$  draw  $FG$ , meeting  $DE$  at  $G$ , making the angle  $DFG$  equal to the angle  $FDG$ ; and from  $F$  draw  $FH$ , meeting  $ED$  at  $H$ , making the angle  $EFH$  equal to the angle  $FEH$ . Then  $FGH$  will be the triangle required.

For  $FG$  is equal to  $DG$ , and  $FH$  is equal to  $HE$ , by I. 6. Therefore the sum of the sides  $FG$ ,  $GH$ ,  $HF$  is equal to  $DE$ , the given perimeter.

Also the angle  $FGH$  is equal to the sum of the angles  $FDG$ ,  $FEH$ , by I. 32; that is to twice the angle  $FDG$ , that is to the angle  $ABC$ . Similarly the angle  $FHG$  is equal to the angle  $ACB$ . Therefore the angle  $GPH$  is equal to the angle  $BAC$ , by I. 32.

## I. 33, 34.

73. Let  $ABCD$  be a quadrilateral having  $AB$  parallel to  $DC$ , and  $AD$  equal to  $BC$ . Suppose  $AB$  less than  $DC$ ; from  $A$  draw a straight line parallel to  $BC$ , meeting  $DC$  at  $E$ . Then  $ABCE$  is a parallelogram. Therefore the angle  $ABC$  is equal to the angle  $AEC$ , and the side  $AE$  is equal to the side  $BC$ , I. 34. Therefore  $AD$  is equal to  $AE$ , and the angle  $ADE$  is equal to the angle  $AED$ . Therefore the angles  $ABC$  and  $ADE$  are equal to the angles  $AEC$  and  $AED$ ; that is they are together equal to two right angles, by I. 13.

In the same manner it may be shewn that the angles  $BAD$  and  $BCD$  are together equal to two right angles.

74. Suppose that  $AB$  and  $CD$  are equal but not parallel, and that the angle  $ABD$  is equal to the angle  $CDB$ . Produce  $AB$  and  $CD$  to meet at  $E$ . Since the angles  $ABD$  and  $CDB$  are equal  $EB$  is equal to  $ED$ . Therefore also  $EA$  is equal to  $EC$ ; and the angle  $ECA$  is equal to the angle  $EAC$ . The two angles  $EBD$  and  $EDB$  are together equal to the two angles  $EAC$  and  $ECA$ , by I. 32. Therefore the angle  $EBD$  is equal to the angle  $EAC$ . Therefore  $BD$  is parallel to  $AC$ , by I. 28.

75. Let  $ABC$  be a triangle; let  $E$  be any point in  $AC$ , and  $D$  any point in  $BC$ : then  $AD$  and  $BE$  will not bisect each other.

Let  $AD$  and  $BE$  intersect at  $F$ .

If possible suppose that  $AF$  is equal to  $FD$ , and  $BF$  equal to  $FE$ . Then the triangles  $AFE$  and  $DFB$  are equal in all respects, by I. 4; so that the angle  $EAF$  is equal to the angle  $BDF$ . Therefore  $AE$  is parallel to  $BD$ , by I. 27. But this is impossible, since  $BD$  and  $AE$ , when produced, meet at  $C$ .

76. Let  $ABCD$  be a quadrilateral, having  $AB$  equal to  $DC$ , and  $AD$  equal to  $BC$ : the figure shall be a parallelogram.

Join  $AC$ ; then in the two triangles  $ABC$  and  $CDA$ , the sides  $BA$ ,  $AC$  are equal to the sides  $DC$ ,  $CA$  each to each; and the base  $BC$  is equal to the base  $DA$ : therefore the angle  $BAC$  is equal to the angle  $DCA$ . Therefore  $AB$  is parallel to  $DC$ , by I. 27.

Similarly it may be shewn that  $AD$  is parallel to  $BC$ .

77. Let  $ABCD$  be a quadrilateral such that the angle  $A$  is equal to the angle  $C$ , and the angle  $B$  equal to the angle  $D$ : then the figure shall be a parallelogram.

The angle  $A$  is equal to the angle  $C$ , and the angle  $B$  is equal to the angle  $D$ ; therefore the two angles  $A$  and  $B$  together are equal to the two angles  $C$  and  $D$  together. But the four angles  $A$ ,  $B$ ,  $C$ ,  $D$  together are equal to four right angles, by I. 32; since the quadrilateral may be divided into two triangles by drawing  $AC$  or  $BD$ . Thus the angles  $A$  and  $B$  are together equal to two right angles. Therefore  $AD$  is parallel to  $BC$ , by I. 28.

Similarly  $AB$  is parallel to  $DC$ .

78. Let  $ABCD$  be a parallelogram. Draw the diagonals  $AC$  and  $BD$  intersecting at  $E$ : then  $AC$  and  $BD$  shall be bisected at  $E$ .

In the triangles  $AED$  and  $CEB$  the sides  $AD$  and  $BC$  are equal, by I. 34; the angles  $ADE$  and  $CBE$  are equal, by I. 29; and the angles  $DAE$  and  $BCE$  are equal, by I. 29. Therefore the triangles are equal in all respects by I. 26. Thus  $AE$  is equal to  $CE$ , and  $DE$  is equal to  $BE$ .

79. Let  $ABCD$  be a quadrilateral figure, and let  $AC$  and  $BD$  intersect at  $E$ ; suppose that  $AC$  and  $BD$  are bisected at  $E$ : then  $ABCD$  will be a parallelogram.

In the triangles  $AED$  and  $CEB$  the sides  $AE, ED$  are equal to the sides  $CE, EB$  each to each, by supposition; the angle  $AED$  is equal to the angle  $CEB$ , by I. 15; therefore the triangles are equal in all respects, by I. 4. Thus the angle  $ADE$  is equal to the angle  $CBE$ ; therefore  $AD$  is parallel to  $CB$ , by I. 27.

Similarly  $AB$  is parallel to  $DC$ .

80. Let  $ABCD$  be a parallelogram, and suppose that the straight line  $AC$  bisects the angles at  $A$  and  $C$ : then the four sides of the parallelogram will be equal.

For in the triangles  $ACB$  and  $ACD$  the side  $AC$  is common, the angle  $ACB$  is equal to the angle  $ACD$ , and the angle  $BAC$  is equal to the angle  $DAC$ ; therefore the side  $BC$  is equal to the side  $DC$ , and the side  $BA$  to the side  $DA$ , by I. 26. But  $AB$  is equal to  $DC$ , and  $AD$  is equal to  $BC$ , by I. 34. Therefore the four sides  $AB, BC, CD, DA$  are all equal.

81. Let  $AB$  and  $CD$  be the parallel straight lines, and  $O$  the given point. In  $AB$  take any point  $E$ , and from  $E$  as centre, with radius equal to the given length, describe a circle meeting  $CD$  at  $F$ ; join  $EF$ . Through  $O$  draw a straight line parallel to  $EF$ , meeting  $AB$  at  $G$ , and  $CD$  at  $H$ . Then  $EGHF$  is a parallelogram, by construction; therefore  $GH$  is equal to  $EF$ , by I. 34: thus  $GH$  is equal to the given length.

82. Let  $ABCD$  be a parallelogram; let straight lines bisecting the angles  $A$  and  $B$  meet at  $E$ : then  $AEB$  will be a right angle.

The angles  $EAB$  and  $EBA$  are together half of the angles  $DAB$  and  $ABC$  together. But the angles  $DAB$  and  $ABC$  are together equal to two right angles, by I. 29. Therefore the angles  $EAB$  and  $EBA$  are together equal to a right angle. Therefore the angle  $AEB$  is a right angle, by I. 32.

83. Let  $ABCD$  be a parallelogram. Suppose that the straight lines which bisect the angles  $A$  and  $C$  are not coincident: then they shall be parallel.

Let the straight line which bisects the angle  $A$  meet  $BC$  at  $E$ ; and let the straight line which bisects the angle  $C$  meet  $DA$  at  $F$ . Then, by I. 29, the angle  $BEA$  is equal to the angle  $DAE$ , that is to the half of the angle  $DAB$ , that is to the half of the angle  $DCB$ , by I. 34. Thus the angle  $BEA$  is equal to the angle  $BCF$ . Therefore  $EA$  is parallel to  $CF$ , by I. 28.

84. Let  $ABCD$  be a parallelogram, and suppose that the diagonals  $AC$  and  $BD$  are equal: then all the angles of the parallelogram will be equal.

In the two triangles  $ABC$  and  $BAD$  the side  $AB$  is common;  $AD$  is equal to  $BC$ , by I. 34; and  $AC$  is equal to  $BD$  by supposition: therefore the angle  $ABC$  is equal to the angle  $BAD$ . Similarly it may be shewn that any other two adjacent angles are equal; so that all the four angles are equal. Hyd

85. Let  $AB$  and  $CD$  be the given straight lines; suppose that the required point is to be at a distance equal to  $E$  from  $AB$ , and at a distance equal to  $F$  from  $CD$ .

Draw a straight line parallel to  $AB$ , and at a distance  $E$  from it; also draw a straight line parallel to  $CD$ , and at a distance  $F$  from it; let the two straight lines thus drawn meet at  $O$ : then  $O$  will be the required point.

For the distance of  $O$  from  $AB$  will be equal to  $E$ , and the distance of  $O$  from  $CD$  will be equal to  $F$ , by I. 84.

Two straight lines can be drawn parallel to  $AB$ , and at the required distance from it, namely, one on each side of it; and in like manner two straight lines can be drawn parallel to  $CD$ , and at the required distance from it: hence four points can be found which will satisfy the conditions of the problem, assuming that  $AB$  and  $CD$  are not parallel.

86. Let  $AB$  and  $CD$  be the two given straight lines in which the required straight line is to be terminated. Let  $E$  be a straight line to which the required straight line is to be equal, and  $F$  that to which the required straight line is to be parallel.

From  $A$  draw a straight line parallel to  $F$ , by I. 81; and cut off from it  $AG$  equal to  $E$ , by I. 3. Through  $G$  draw a straight line parallel to  $AB$ , and let it meet  $CD$  at  $H$ . Through  $H$  draw a straight line parallel to  $AG$ , and let it meet  $AB$  at  $K$ . Then  $HK$  is the required straight line.

For  $HK$  is equal to  $AG$ , by I. 84; so that  $HK$  is of the required length: and it is parallel to  $AG$ , and therefore to  $F$ , by I. 80.

87. Let  $AEB$ ,  $BFC$ ,  $CGD$  be the three equilateral triangles: then will  $EF$  be equal to  $AC$ , and  $GF$  be equal to  $BD$ .

In the two triangles  $ABC$  and  $EBF$  the two sides  $AB$ ,  $BC$  are equal to the two sides  $EB$ ,  $BF$  each to each. The angle  $FBC$  is equal to the angle  $ABE$ , each being the angle of an equilateral triangle; to each of them add the angle  $ABF$ ; therefore the angle  $ABC$  is equal to the angle  $EBF$ . Hence the triangles  $ABC$  and  $EBF$  are equal in all respects, by I. 4; so that  $AC$  is equal to  $EF$ .

Similarly it may be shewn that  $BD$  is equal to  $GF$ .

88. Let  $ABCD$  be a parallelogram, and let  $ABEF$  be another parallelogram having  $BE$  equal to  $BC$ , but the angle  $ABE$  greater than the angle  $ABC$ : then will the diagonal  $BF$  be less than the diagonal  $BD$ .

The two angles  $ABC$ ,  $BCD$  are together equal to two right angles, and so also are the two angles  $ABE$ ,  $BEF$ , by I. 29; therefore the two angles  $ABC$ ,  $BCD$  are together equal to the two  $ABE$ ,  $BEF$ : but the angle  $ABE$  is greater than the angle  $ABC$ , by hypothesis; therefore the angle  $BEF$  is less than the angle  $BCD$ .

In the two triangles  $BCD$ ,  $BEF$  the two sides  $BC$ ,  $CD$  are equal to the two sides  $BE$ ,  $EF$  each to each; but the angle  $BCD$  is greater than the angle  $BEF$ : therefore the base  $BD$  is greater than the base  $BF$ , by I. 24.

89. Let  $AD$ ,  $BE$ ,  $CF$  be the perpendiculars from  $A$ ,  $B$ ,  $C$  respectively on the straight line: the sum of  $AD$  and  $CF$  shall be equal to twice  $BE$ .

The straight line  $DF$  produced does not pass between  $A$  and  $C$ ; suppose that it cuts  $AC$  produced through  $C$ . Through  $E$  draw a straight line parallel to  $AC$ , and let it meet  $AD$  at  $G$ , and  $CF$  produced, through  $F$ , at  $H$ . Thus  $GE$  is equal to  $AB$ , and  $EH$  is equal to  $BC$ , by I. 34; therefore  $GE$  is equal to  $EH$ .

In the two triangles  $GED$ ,  $HEF$  the angle  $GED$  is equal to the angle  $HEF$ , by I. 15; the angle  $DGE$  is equal to the angle  $FHE$ , by I. 29; and the side  $GE$  was shewn equal to the side  $HE$ : hence these triangles are equal in all respects, by I. 26; so that  $GD$  is equal to  $HF$ . Therefore  $AD$  and  $CF$  together are equal to  $AG$  and  $CH$  together, that is equal to twice  $BE$ , by I. 34.

90. Let  $ABCD$  be the parallelogram; let  $O$  be the point of intersection of the diagonals  $AC$  and  $BD$ . Then by Exercise 78 the diagonals  $AC$  and  $BD$  are bisected at  $O$ . By Exercise 89 the sum of the perpendiculars from  $A$  and  $C$  on any straight line outside the parallelogram is twice the perpendicular from  $O$ ; and also the sum of the perpendiculars from  $B$  and  $D$  is twice the perpendicular from  $O$ . Hence the sum of the perpendiculars from  $A$  and  $C$  is equal to the sum of the perpendiculars from  $B$  and  $D$ .

91. Let  $ABCDEF$  be the six-sided figure. Then  $AB$  is by supposition equal and parallel to  $ED$ ; therefore  $ABDE$  is a parallelogram, by I. 33. Therefore  $AD$  passes through the middle point of  $BE$ , by Exercise 78. Similarly it can be shewn that  $CF$  passes through the middle point of  $BE$ . Thus  $AD$ ,  $BE$ , and  $CF$  meet at a point.

92. Through  $E$  draw a straight line parallel to  $AB$ , and let it meet  $AC$  at  $F$ . On  $FC$  take  $FG$  equal to  $AF$ . Join  $GE$  and produce it to meet  $AB$  at  $H$ : then  $GEH$  shall be the straight line required.

Through  $F$  draw a straight line parallel to  $GH$ , and let it meet  $AB$  at  $K$ . In the triangles  $AFK$  and  $FGE$  the side  $AF$  is equal to the side  $FG$  by construction; the angle  $AFK$  is equal to the angle  $FGE$ , and the angle  $FAK$  is equal to the angle  $GFE$ , by I. 29: therefore  $FK$  is equal to  $GE$ , by I. 26. But  $FK$  is equal to  $EH$ , by I. 34: therefore  $GE$  is equal to  $EH$ , so that  $GH$  is bisected at  $E$ .

93. Let  $ABCD$  be the given parallelogram, and  $P$  the given point on the side  $AB$ . On  $CD$  take  $CQ$  equal to  $AP$ ; join  $AC$  and  $PQ$  intersecting at  $R$ . Through  $R$  draw a straight line at right angles to  $PQ$ , meeting  $AD$  at  $S$ , and  $CB$  at  $T$ . Then  $PSQT$  will be the required rhombus.

In the triangles  $APR$  and  $CQR$  the sides  $AP$  and  $CQ$  are equal, by construction; the angle  $ARP$  is equal to the angle  $CRQ$ , by I. 15; and the angle  $RAP$  is equal to the angle  $RCQ$ , by I. 29: therefore  $PR$  is equal to  $QR$ , and  $AR$  is equal to  $CR$ , by I. 26.

In the triangles  $PRS$  and  $QRS$  the sides  $PR$  and  $QR$  are equal;  $RS$  is common; and the angles  $PRS$  and  $QRS$  are equal being right angles: therefore  $PS$  is equal to  $QS$ , by I. 4.

In the triangles  $CRT$  and  $ARS$  the sides  $AR$  and  $CR$  are equal; the angle  $ARS$  is equal to the angle  $CRT$ , by I. 15; and the angle  $ASR$  is equal to the angle  $CTR$ , by I. 29: therefore  $RS$  is equal to  $RT$ , by I. 26.

In the triangles  $SRP$  and  $TRP$  the sides  $SR$  and  $TR$  are equal; the side  $RP$  is common; and the angles  $SRP$  and  $TRP$  are equal being right angles: therefore  $SP$  is equal to  $TP$ , by I. 4.

In the same manner it may be shewn that  $TQ$  is equal to  $SQ$ , and also equal to  $TP$ . Hence  $PSQT$  is a rhombus.

The construction fails if the straight line through  $R$  at right angles to  $PQ$ , instead of meeting  $AD$  and  $CB$ , meets  $DC$  and  $BA$ .

94. Let  $BE$  intersect  $AC$  at  $G$ , and  $DF$  intersect  $AC$  at  $H$ . Through  $G$  draw a straight line parallel to  $AD$ , meeting  $DF$  at  $K$ .

Then  $ED$  is equal and parallel to  $BF$ ; therefore  $EB$  is equal and parallel to  $DF$ , so that  $EGKD$  is a parallelogram, and  $GK$  is equal to  $ED$ , and therefore equal to  $AE$ . In the triangles  $AEF$  and  $GKH$  the sides  $AE$  and  $GK$  are equal; the angles  $EAG$  and  $KGH$  are equal, and the angles  $EGA$  and  $KHG$  are equal, by I. 29: therefore  $AG$  is equal to  $GH$ , by I. 26.

Similarly it may be shewn that  $CH$  is equal to  $HG$ : hence the three straight lines  $AG$ ,  $GH$ ,  $HC$  are all equal; so that  $AC$  is trisected.

#### I. 35 to 45.

95. Let  $O$  be the middle point of  $DC$ . Of the two straight lines  $AD$  and  $BC$ , suppose  $AD$  the less. Through  $O$  draw a straight line parallel to  $AB$ , meeting  $AD$  produced at  $E$ , and meeting  $BC$  at  $F$ .

Then in the two triangles  $EOD$  and  $FOC$  the sides  $DO$  and  $CO$  are equal; the angle  $EOD$  is equal to the angle  $FOC$ , by I. 15; and the angle  $OED$  is equal to the angle  $OFC$ , by I. 29: therefore the triangles are equal by I. 26. To each triangle add the figure  $ADOFB$ : thus the figure  $ABFE$  is equal to the figure  $ABCD$ .

96. Construct a parallelogram by drawing through  $E$  a straight line parallel to  $AB$ ; this parallelogram is equal to  $ABCD$  by Exercise 95. The triangle  $AEB$  is half this parallelogram, by I. 41: therefore the triangle  $AEB$  is half the quadrilateral  $ABCD$ .

97. Let  $ABCD$  be a parallelogram; let  $O$  be the middle point of the diagonal  $AC$ ; through  $O$  draw any straight line meeting  $AB$  at  $E$  and  $CD$  at  $F$ : then the straight line  $EOF$  shall bisect the parallelogram.

In the two triangles  $AOE$  and  $COF$  the sides  $AO$  and  $CO$  are equal; the angle  $AOE$  is equal to the angle  $COF$ , by I. 15; and the angle  $OAE$  is equal to the angle  $OCF$ , by I. 29: therefore the triangles are equal, by I. 26. To each triangle add the figure  $AOFD$ ; thus the figure  $Aefd$  is equal to the triangle  $ACD$ . But the triangle  $ACD$  is half the parallelogram  $ABCD$ : therefore the figure  $Aefd$  is half the parallelogram  $ABCD$ .

98. Let  $ABCD$  be the parallelogram, and  $P$  the given point within it. Bisect  $AC$  at  $O$ ; join  $PO$  and produce it to meet opposite sides of the parallelogram at  $E$  and  $F$  respectively. Then, by Exercise 97, the straight line  $EPF$  bisects the parallelogram.

99. Let  $ABCD$  be the parallelogram. Bisect  $AC$  at  $O$ ; through  $O$  draw a straight line at right angles to  $AC$ , and let it intersect at  $E$  the straight line drawn through  $D$  parallel to  $AC$ . Produce  $EO$  through  $O$  to  $F$ , making  $OF$  equal to  $OE$ . Then  $AFCE$  will be such a rhombus as is required.

For in the two triangles  $AOE$  and  $COE$  the two sides  $AO$ ,  $OE$  are equal to the two sides  $CO$ ,  $OE$  each to each; and the right angles  $AOE$ ,  $COE$  are equal: therefore  $AE$  is equal to  $CE$ . Similarly we can shew that  $AF$  is equal to  $AE$ ; and also that  $CF$  is equal to  $CE$ , and to  $AF$ . Therefore  $AFCE$  is a rhombus. Also the triangle  $AEC$  is equal to the triangle  $ADC$ , by I. 37: therefore the rhombus  $AFCE$  is equal to the parallelogram  $ABCD$ .

100. Let  $ABC$ ,  $DEF$  be two triangles having the sides  $AB$ ,  $BC$  equal to the sides  $DE$ ,  $EF$  each to each; also the angles  $ABC$  and  $DEF$  together equal to two right angles: then the triangles shall be equal in area.

Produce  $CB$  to  $G$ , making  $BG$  equal to  $BC$  or  $EF$ ; and join  $AG$ . Then in the two triangles  $ABG$ ,  $DEF$  the sides  $AB$ ,  $BG$  are equal to the sides  $DE$ ,  $EF$  each to each; also the angles  $ABC$  and  $DEF$  are equal to two right angles by hypothesis, and the angles  $ABC$  and  $ABG$  are equal to two right angles by I. 13; therefore the angle  $ABG$  is equal to the angle  $DEF$ . Hence the two triangles  $ABG$ ,  $DEF$  are equal in all respects, by I. 4.

Now the triangles  $ABC$  and  $ABG$  are equal in area by I. 38; therefore the triangles  $ABC$  and  $DEF$  are equal in area.

101. The triangle  $BEC$  is half the parallelogram  $ABCD$  by I. 41; therefore the triangle  $BEC$  is equal to the figure  ~~$ABEC$~~ . Take away the triangle  $FEC$  from both; then the remainders are equal; that is the triangle  $EBF$  is equal in area to the triangle  $CED$ .  $FEDC$

102. Let  $ABCD$  be a parallelogram; and let  $AC$  and  $BD$  intersect at  $O$ . Then, by Exercise 78, the straight lines  $AC$  and  $BD$  are bisected at  $O$ . Because  $AO$  is equal to  $OC$  the triangles  $AOB$  and  $COB$  are equal, by I. 38. Similarly  $BOC$  and  $DOC$  are equal; also  $COD$  and  $AOD$  are equal. Thus the four triangles are all equal.

103. The triangle  $AEC$  is equal to the triangle  $BED$ ; to each of these add the triangle  $BEC$ ; therefore the triangle  $ABC$  is equal to the triangle  $DBC$ ; therefore  $DA$  is parallel to  $BC$ , by I. 39.

104. Let the diagonals intersect at  $O$ ; then  $AO$  is equal to  $CO$  by Exercise 78. The triangle  $AOB$  is equal to the triangle  $COB$ , and the triangle  $AOP$  is equal to the triangle  $COP$ , by I. 38. Therefore the triangle  $PAB$  is equal to the triangle  $PCB$ .

105. Let  $ABCD$  be any quadrilateral figure. Through  $A$  and  $C$  draw straight lines parallel to  $BD$ , and through  $B$  and  $D$  draw straight lines parallel to  $AC$ : thus a parallelogram is formed having two opposite sides equal to  $BD$ , and two opposite sides equal to  $AC$ , and its angles equal to those at  $O$  the intersection of  $AC$  and  $BD$ . Also this parallelogram is double the figure  $ABCD$ ; for it consists of four parallelograms which are double of  $AOB$ ,  $BOC$ ,  $COD$ ,  $DOA$  respectively. By drawing a diagonal of this parallelogram we obtain two triangles having two sides equal to the diagonals of the given quadrilateral, and an angle equal to one of those at  $O$ ; and this triangle will be equal in area to the given quadrilateral by I. 41.

106. Let  $ABC$  be any triangle; let  $D$  be the middle point of  $BC$ , and  $E$  the middle point of  $AC$ : then  $ED$  shall be parallel to  $AB$ .

Join  $AD$  and  $BE$ . The triangle  $AED$  is equal to the triangle  $CED$ , by I. 38. The triangle  $BED$  is equal to the triangle  $CED$ , by I. 38. Therefore the triangle  $AED$  is equal to the triangle  $BED$ . Therefore  $AB$  is parallel to  $ED$ , by I. 39.

107. Let  $ABCD$  be a quadrilateral; let  $E$ ,  $F$ ,  $G$ ,  $H$  be the middle points of  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  respectively: then shall  $EFGH$  be a parallelogram.

$EF$  and  $GH$  are both parallel to  $AC$ , by Exercise 106; therefore they are parallel to each other, by I. 30. Similarly  $FG$  and  $HE$  are parallel: therefore  $EFGH$  is a parallelogram.

108. The triangle  $BDC$  is equal to the triangle  $ABE$ , for each of them is half the triangle  $ABC$ , by I. 38. Take away the triangle  $BDF$ , and the



remainders are equal; that is the triangle  $BFC$  is equal to the quadrilateral  $ADFE$ .

109. Let  $ABC$  be a triangle; let  $D$  be the middle point of  $CB$ , and  $E$  the middle point of  $CA$ :  $ED$  shall be equal to half of  $AB$ .

Through  $D$  draw a straight line parallel to  $CA$ , and let it meet  $AB$  at  $F$ ; then  $ED$  is parallel to  $AF$ , by Exercise 106: therefore  $AF$  is equal to  $ED$ , by I. 34.

In the two triangles  $CDE$ ,  $DBF$  the sides  $CD$ ,  $DB$  are equal by hypothesis; the angle  $CDE$  is equal to the angle  $DBF$ , and the angle  $ECD$  is equal to the angle  $FDB$ , by I. 29: therefore  $ED$  is equal to  $FB$ , by I. 26. Therefore  $AF$  is equal to  $FB$ ; therefore  $AF$  is half of  $AB$ ; therefore  $ED$  is half of  $AB$ .

110. By Exercise 106 the straight lines  $EG$  and  $FH$  are both parallel to  $BD$ , therefore they are parallel to each other, by I. 30; and by Exercise 109 they are each equal to half of  $BD$ , and therefore they are equal to each other.

111. Let  $D$ ,  $E$ ,  $F$  be the three given points; through  $D$  draw a straight line parallel to  $EF$ , through  $E$  draw a straight line parallel to  $FD$ , and through  $F$  draw a straight line parallel to  $DE$ . Let the first and second straight line meet at  $C$ , the second and third at  $A$ , and the third and first at  $B$ . Then  $ABC$  shall be the triangle required.

For by construction  $BDEF$  and  $DCEF$  are parallelograms, so that  $BD$  and  $DC$  are each equal to  $FE$ , by I. 34: therefore  $BD$  is equal to  $DC$ , and  $D$  is the middle point of  $BC$ . Similarly  $E$  is the middle point of  $CA$ , and  $F$  is the middle point of  $AB$ .

112. Let  $ABC$  be any triangle; let  $D$  be the middle point of  $BC$ , and  $E$  the middle point of  $CA$ : then the triangle  $EDC$  shall be one-fourth of the triangle  $ABC$ .

Join  $EB$ . The triangle  $EBA$  is equal to the triangle  $EBC$  by I. 38; therefore the triangle  $EBC$  is half the triangle  $ABC$ . Again, the triangle  $DEC$  is equal to the triangle  $BED$  by I. 38; therefore the triangle  $DEC$  is half the triangle  $BEC$ ; therefore the triangle  $DEC$  is one-fourth the triangle  $ABC$ .

113.  $EA$  is equal to  $ED$ , by Exercise 59; therefore the angle  $EAD$  is equal to the angle  $EDA$ . Similarly the angle  $FAD$  is equal to the angle  $FDA$ . Therefore the whole angle  $EAF$  is equal to the whole angle  $EDF$ .

Again, the triangle  $EAD$  is equal to the triangle  $EBD$ , and the triangle  $FAD$  is equal to the triangle  $FCD$ , by I. 38. Therefore  $AFDE$  is equal to  $EBD$  and  $FCD$  together. Therefore  $AFDE$  is half the triangle  $ABC$ .

114. Let  $ABC$ ,  $DBC$  be triangles of equal area on opposite sides of the same base  $BC$ ; join  $AD$  cutting  $BC$ , or  $BC$  produced, at  $F$ : then shall  $AF$  be equal to  $FD$ .

Make the triangle  $BEC$  on the same side of  $BC$  as  $BDC$ , so that  $BE$  may be equal to  $BA$  and  $CE$  to  $CA$ ; then the triangle  $BCE$  is equal to the triangle  $BCA$  in all respects, by I. 8. Therefore the triangles  $BCE$  and  $BDC$  are equal in area. Therefore  $ED$  is parallel to  $BC$ , by I. 39. Therefore the triangle  $FCE$  is equal to the triangle  $FCD$ , by I. 37.

In the two triangles  $CAF$  and  $CEF$  the side  $CF$  is common; the sides  $CA$ ,  $CE$  are equal by construction; and the angles  $ACF$  and  $ECF$  are equal,

since the angles  $ACB$  and  $ECB$  are equal: hence these triangles are equal in all respects, by I. 4.

Thus the area of the triangle  $ACF$  is equal to the area of the triangle  $ECF$ ; and therefore the triangles  $ACF$  and  $DCF$  are equal in area. Thus  $AF$  must be equal to  $DF$ ; for if these are not equal it can be shewn by the aid of I. 38, that the areas are not equal.

115. Let  $ABCD$ ,  $BEFC$ ,  $EGHF$  be the three parallelograms. Join  $AF$  cutting  $BC$  at  $K$ ; join  $BH$  cutting  $EF$  at  $L$ : then  $BLFK$  shall be half of  $BEFC$ .

$AF$  is parallel to  $BH$ , by I. 33. In the triangles  $ABK$ ,  $BEL$  the sides  $AB$  and  $BE$  are equal, by hypothesis; the angles  $ABK$  and  $BEL$ , and the angles  $BAK$  and  $EBL$  are equal, by I. 29: therefore the triangles  $ABK$  and  $BEL$  are equal, and  $BK$  is equal to  $EL$ , by I. 26.

Thus  $EL$  is equal to  $LF$ , and the triangle  $ELB$  is half the parallelogram  $LFBK$ , by I. 38 and I. 41.

In the same way it may be shewn that the triangles  $ABK$  and  $FCK$  are equal, so that  $BK$  is equal to  $CK$ , and the triangle  $CKF$  is half the parallelogram  $KBLF$ .

Hence the triangles  $BEL$  and  $FCK$  are together equal to the parallelogram  $KBLF$ ; so that  $KBLF$  is half  $BEFC$ .

116. The triangle  $BCG$  is equal to the triangle  $BDG$ , by I. 37. Also the triangle  $BDG$  is equal to the sum of  $BFG$  and  $BFD$ , that is to the sum of  $BFG$  and  $BFA$ , by I. 37. And the triangle  $BCG$  is the sum of  $BFG$  and  $CFG$ . Hence the sum of  $BFG$  and  $BFA$  is equal to the sum of  $BFG$  and  $CFG$ . Therefore the triangle  $BFA$  is equal to the triangle  $CFG$ .

117. Suppose  $AD$  greater than  $AB$ . Join  $CD$ ; through  $B$  draw a straight line parallel to  $DC$ , meeting  $AC$  at  $E$ ; join  $ED$ : then  $AED$  will be the triangle required.

For the triangle  $EBD$  is equal to the triangle  $EBC$ , by I. 37. To each of these add the triangle  $ABE$ ; then the triangle  $ADE$  is equal to the triangle  $ABC$ .

If  $AD$  is less than  $AB$  the straight line drawn through  $B$  parallel to  $DC$  will meet  $AC$  produced; but the demonstration will not be essentially changed.

118. Let  $D$  be the given point in  $BC$ . Join  $DA$ ; through  $C$  draw a straight line parallel to  $DA$ , meeting  $BA$  produced at  $E$ ; join  $DE$ : then  $DEB$  will be the triangle required.

For the triangle  $DEA$  is equal to the triangle  $DCA$ , by I. 37. To each of these add the triangle  $ADB$ : then the triangle  $EDB$  is equal to the triangle  $ABC$ .

If the given point  $D$  is in  $BC$  produced the process will not be essentially changed.

119. Let  $P$  be the given point in  $CD$ ; join  $PA$  and  $PB$ . Through  $C$  draw a straight line parallel to  $PB$ , and through  $D$  draw a straight line parallel to  $PA$ . Through  $P$  draw a straight line parallel to  $AB$ , let it meet the straight line drawn through  $C$  at  $E$ , and the straight line drawn through  $D$  at  $F$ : then  $ABEF$  is the quadrilateral required.

For the triangle  $PEB$  is equal to the triangle  $PCB$ , and the triangle  $PFA$  is equal to the triangle  $PDA$ , by I. 37. Therefore the figure  $ABEF$  is equal to  $ABCD$ .

120. Join  $PA$  and  $PB$ . Through  $C$  draw a straight line parallel to  $PB$ , and let it meet  $AB$  produced at  $M$ ; through  $D$  draw a straight line parallel to  $PA$ , and let it meet  $BA$  produced at  $N$ : then  $PMN$  is the triangle required.

For the triangle  $PBC$  is equal to the triangle  $PBM$ , and the triangle  $PAD$  is equal to the triangle  $PAN$ , by I. 37. Therefore the triangle  $PMN$  is equal to  $ABCD$ .

121. Let  $AC$ , produced if necessary, meet the given straight line at  $D$ ; join  $DB$ ; through  $C$  draw a straight line parallel to  $DB$ , meeting  $AB$ , produced if necessary, at  $E$ : then  $AED$  is such a triangle as is required.

For the triangle  $CED$  is equal to the triangle  $CEB$ , by I. 37: therefore the triangle  $AED$  is equal to the triangle  $ABC$ .

122. Let  $ABC$  be the given triangle,  $P$  the given point in the side  $AC$ . Suppose  $P$  to be nearer to  $A$  than to  $C$ . Bisect  $BC$  at  $D$ ; join  $AD$  and  $PD$ . Through  $A$  draw a straight line parallel to  $PD$ , meeting  $BC$  at  $E$ . Join  $EP$ : then  $EP$  will bisect the triangle  $ABC$ .

For the triangle  $PED$  is equal to the triangle  $PAD$ , by I. 37; to each of these add the triangle  $PDC$ : therefore the triangle  $PCE$  is equal to the triangle  $ACD$ . But the triangle  $ACD$  is half the triangle  $ABC$  by I. 38: therefore the triangle  $PEC$  is half the triangle  $ABC$ .

If  $P$  be nearer to  $C$  than to  $A$  the side  $AB$  must be bisected.

123. Let  $ABCD$  be the given quadrilateral,  $A$  the given angular point. Draw the diagonals  $AC$  and  $BD$ ; bisect  $BD$  at  $E$ ; join  $AE$ ,  $CE$ . Through  $E$  draw a straight line parallel to  $AC$ ; suppose this straight line to be further from  $B$  than  $AC$  is, and let it meet  $DC$  at  $G$ . Join  $AG$ : then  $AG$  will bisect the quadrilateral.

The triangle  $AEC$  is equal to the triangle  $AGC$ , by I. 37. To each add the triangle  $ABC$ ; therefore the figure  $ABCE$  is equal to the figure  $ABCG$ .

But the triangle  $ABE$  is half the triangle  $ABD$ , and the triangle  $CBE$  is half the triangle  $CBD$ , by I. 38: therefore the figure  $ABCE$  is half the figure  $ABCD$ . Therefore the figure  $ABCG$  is half the figure  $ABCD$ .

If the straight line drawn through  $E$  parallel to  $AC$  is nearer to  $B$  than  $AC$  is, it will meet  $BC$  instead of  $DC$ ; but the demonstration will not be essentially changed.

124. If possible suppose that  $O$  is not in the diagonal  $AC$ . Let the straight line through  $O$  parallel to  $BC$  meet  $AB$  at  $E$ , and  $DC$  at  $F$ ; and suppose that  $AC$  intersects  $EF$  at a point  $G$  between  $O$  and  $F$ . Through  $G$  draw a straight line parallel to  $AB$ .

Then the parallelogram  $GD$  is equal to the parallelogram  $GB$ , by I. 43. Therefore the parallelogram  $OD$  is greater than the parallelogram  $OB$ . But this is impossible, for they are equal by hypothesis. Therefore the point  $O$  cannot fall otherwise than on  $AC$ .

#### I. 46 to 48.

125. The angles  $ACD$  and  $BCF$  are equal, being right angles; to each add the angle  $ACB$ ; thus the whole angle  $BCD$  is equal to the whole angle  $ACF$ .

In the two triangles  $BCD$  and  $ACF$  the two sides  $BC$ ,  $CD$  are equal to the two sides  $FC$ ,  $CA$  each to each; and the angle  $BCD$  is equal to the angle  $FCA$ : therefore  $FA$  is equal to  $BD$ , by I. 4.

126. Let  $BAC$  be a triangle, having the angle  $BAC$  *acute*; then will the square on  $BC$  be *less* than the sum of the squares on  $BA$  and  $AC$ .

From  $A$  draw a straight line at right angles to  $BA$ , and cut off  $AD$  equal to  $AC$ ; join  $BD$ .

Then  $BD$  is greater than  $BC$  by I. 24. Now the square on  $BD$  is equal to the squares on  $BA$  and  $AD$  by I. 47. Hence the square on  $BC$  is less than the squares on  $BA$  and  $AD$ , that is less than the squares on  $BA$ , and  $AC$ .

127. Let  $BAC$  be a triangle, having the angle  $BAC$  *obtuse*: then will the square on  $BC$  be greater than the sum of the squares on  $BA$  and  $AC$ .

From  $A$  draw a straight line at right angles to  $BA$ , and cut off  $AD$  equal to  $AC$ ; join  $BD$ .

Then  $BD$  is less than  $BC$ , by I. 24. Now the square on  $BD$  is equal to the squares on  $BA$  and  $AD$ , by I. 47. Hence the square on  $BC$  is greater than the squares on  $BA$  and  $AD$ , that is greater than the squares on  $BA$  and  $AC$ .

128. Let  $ABC$  be a triangle; and suppose the square on  $BC$  *less* than the squares on  $BA$  and  $AC$ : then the angle  $BAC$  will be an *acute* angle.

The angle  $BAC$  cannot be a right angle, for then the square on  $BC$  would be equal to the squares on  $BA$  and  $AC$ , by I. 47. The angle  $BAC$  cannot be obtuse, for then the square on  $BC$  would be greater than the squares on  $BA$  and  $AC$ , by Exercise 127. Therefore the angle  $BAC$  must be an acute angle.

Again, let  $ABC$  be a triangle; and suppose the square on  $BC$  *greater* than the squares on  $BA$  and  $AC$ : then the angle  $BAC$  will be an *obtuse* angle.

The angle  $BAC$  cannot be a right angle, for then the square on  $BC$  would be equal to the squares on  $BA$  and  $AC$ , by I. 47. The angle  $BAC$  cannot be acute, for then the square on  $BC$  would be less than the squares on  $BA$  and  $AC$ , by Exercise 126. Therefore the angle  $BAC$  must be an obtuse angle.

129. Let  $BAC$  be a triangle, having the angle  $A$  a right angle. Let a straight line meet  $AB$  at  $D$ , and  $AC$  at  $E$ : then shall the squares on  $BE$  and  $CD$  be equal to the sum of the squares on  $BC$  and  $DE$ .

The square on  $BE$  is equal to the squares on  $BA$  and  $AE$ , and the square on  $CD$  is equal to the squares on  $CA$  and  $AD$ , by I. 47. Therefore the squares on  $BE$  and  $CD$  are equal to the squares on  $BA$ ,  $CA$ ,  $AE$ ,  $AD$ ; that is to the squares on  $BC$ ,  $AE$ ,  $AD$ , that is to the squares on  $BC$  and  $DE$ .

130. Draw through  $P$  a straight line parallel to  $AD$ , meeting  $AB$  at  $K$  and  $CD$  at  $L$ . Draw through  $P$  a straight line parallel to  $AB$ , meeting  $BC$  at  $M$  and  $AD$  at  $N$ . Then  $AK$  is equal to  $DL$ , and  $KB$  is equal to  $LC$ , by I. 34.

The squares on  $PA$  and  $PC$  are together equal to the squares on  $AK$ ,  $PK$ ,  $CL$ ,  $LP$ , by I. 47; that is to the squares on  $DL$ ,  $PK$ ,  $KB$ ,  $LP$ ; that is

to the squares on  $DL$ ,  $LP$ ,  $PK$ ,  $KB$ ; that is to the squares on  $PD$  and  $PB$ , by I. 47.

131. Let  $ABC$  be a triangle having a right angle at  $C$ ; and let the square on  $AC$  be three times the square on  $BC$ . From  $C$  draw  $CD$  to bisect  $AB$ , and  $CE$  perpendicular to  $AB$ . Then will the angles  $ACD$ ,  $DCE$ ,  $ECB$  be all equal.

The square on  $AB$  is equal to the squares on  $AC$  and  $BC$ , that is to three times the square on  $BC$  and the square on  $BC$ , that is to four times the square on  $BC$ . Hence it may be shewn that  $AB$  is twice  $BC$ . But  $AB$  is twice  $DC$ , by Exercise 59. Thus  $BC$ ,  $CD$ ,  $DB$  are all equal, so that  $BCD$  is an equilateral triangle. Hence the angle  $BCD$  is two-thirds of a right angle, and  $ACD$  is one-third of a right angle, by I. 32.

Again, in the two triangles  $BEC$  and  $DEC$ , the sides  $BC$  and  $DC$  are equal; therefore the angle  $CDB$  is equal to the angle  $CBD$ ; and the angles  $BEC$  and  $DEC$  are equal being right angles: therefore the angles  $BCE$  and  $DCE$  are equal, by I. 32. Hence  $BCE$  and  $DCE$  are each one-third of a right angle; so that the three angles  $ACD$ ,  $DCE$ ,  $BCE$  are all equal.

132. Let  $ABC$  be a triangle, having a right angle at  $A$ ; let  $E$  be the middle point of  $AC$ , and  $F$  the middle point of  $AB$ : then four times the squares on  $BE$  and  $CF$  will be equal to five times the square on  $BC$ .

For four times the square on  $BE$  is equal to four times the square on  $AB$  and four times the square on  $AE$ , by I. 47. And four times the square on  $CF$  is equal to four times the square on  $AC$  and four times the square on  $AF$ . Therefore four times the squares on  $BE$  and  $CF$  are equal to four times the square on  $AE$ , four times the square on  $AF$ , and four times the squares on  $AB$  and  $AC$ ; that is to four times the square on  $AE$ , four times the square on  $AF$ , and four times the square on  $BC$ ; that is to the squares on  $AC$  and  $AB$ , and four times the square on  $BC$ ; that is to five times the square on  $BC$ .

133. From  $D$  draw a perpendicular  $DM$  on  $GB$  produced. The angles  $DBM$  and  $MBC$  are together equal to a right angle; and so are the angles  $CBA$  and  $MBC$ : hence the angles  $DBM$  and  $CBA$  are equal.

In the two triangles  $DBM$ ,  $CBA$  the sides  $DB$  and  $CB$  are equal; the angles  $DBM$  and  $CBA$  are equal; and the right angles  $DMB$  and  $CAB$  are equal: hence  $BM$  is equal to  $BA$ , and  $DM$  is equal to  $CA$ , by I. 26. Therefore  $GM$  is equal to twice  $BA$ , and the square on  $GM$  is equal four times the square on  $BA$ . The square on  $DG$  is equal to the squares on  $GM$  and  $DM$ , by I. 47; that is to four times the square on  $BA$  and the square on  $CA$ . Similarly it may be shewn that the square on  $EF$  is equal to four times the square on  $CA$  and the square on  $BA$ . Hence the squares on  $DG$  and  $EF$  are equal to five times the square on  $BA$  and five times the square on  $CA$ , that is to five times the square on  $BC$ , by I. 47.

## II. 1 to 11.

134. Let a straight line  $AB$  be divided into two parts at  $C$ , and suppose that the squares on  $AC$  and  $CB$  are equal to twice the rectangle  $AC \cdot CB$ : then shall  $AC$  be equal to  $CB$ .

For if  $AC$  be not equal to  $CB$  suppose  $AC$  the greater, and construct the diagram of II. 4; on  $CA$  take  $CX$  equal to  $CB$ , draw through  $X$  a straight

line parallel to  $AD$ , meeting  $HG$  at  $Y$  and  $DF$  at  $Z$ . Then we have given that  $HF$  and  $CK$  are together equal to  $AG$  and  $GE$ . By our construction we make  $XG$  equal to  $CK$ , and  $YF$  equal to  $GE$ . Thus  $HZ$  and  $CK$  are together equal to  $AG$ ; therefore  $HZ$  is equal to  $AY$ . But this is impossible for  $HD$  is greater than  $AH$ . Hence  $AC$  and  $CB$  cannot be unequal; that is they are equal.

135. By II. 5 the rectangle contained by the parts is always *less* than the square on half the line, except when the straight line is bisected; so that the rectangle contained by the parts is *greatest* when the straight line is bisected.

136. Take  $AC$  equal to a side of the smaller square; produce  $AC$  to  $D$  so that  $CD$  is equal to a side of the larger square: and from  $CD$  cut off  $CB$  equal to  $CA$ . Then by II. 6 the rectangle  $AD$ ,  $DB$  is equal to the difference of the squares on  $CD$  and  $CB$ . Thus the required rectangle is found.

137. By II. 9 the sum of the squares on the two parts is always *greater* than double the square on half the straight line, except when the straight line is bisected; so that the sum of the squares on the parts is *least* when the straight line is bisected.

138. Take  $AC$  equal to the greater of the two straight lines; on  $AC$  produced take  $CD$  equal to the less of the two straight lines, and also take  $CB$  equal to  $AC$ . Then  $AD$  is equal to the sum of the two straight lines, and  $DB$  is equal to their difference. And it is shewn in II. 9 that the squares on  $AD$  and  $DB$  are together double of the squares on  $AC$  and  $CD$ .

139. Let  $AB$  be the given straight line to be divided;  $KL$  a side of the given square.

Make the angle  $ABE$  equal to half a right angle. With centre  $A$  and radius equal to  $KL$  describe a circle cutting  $BE$  at  $F$ . From  $F$  draw  $FD$  perpendicular to  $AB$ : then  $AB$  shall be divided at  $D$  in the manner required.

For it may be shewn as in II. 9 that  $FD$  is equal to  $DB$ . Also the squares on  $AD$  and  $DF$  are equal to the square on ~~AK~~, so that the squares on  $AD$  and  $DB$  are equal to the square on  $KL$ .  $\triangle F$

A remark may be made like the last sentence of the solution of Exercise 65.  $\triangle F$  is right

140. Let  $AB$  be the given straight line. Produce  $AB$  to  $C$  so that  $AC$  may be equal to the diagonal of the square described on  $AB$ ; and from  $BA$  cut off  $BD$  equal to  $BC$ : then will the square on  $DA$  be double the square on  $DB$ .

Since the straight line  $CD$  is bisected at  $B$  and produced to  $A$ , we have the squares on  $CA$  and  $DA$  together equal to double the squares on  $BA$  and  $BD$ , by II. 10. But the square on  $CA$  is double the square on  $BA$ , by I. 47: therefore the square on  $DA$  is double the square on  $DB$ .

141. In the triangles  $HAC$ ,  $FAB$  the two sides  $HA$ ,  $AC$  are equal to the two sides  $FA$ ,  $AB$  each to each; and the right angles  $HAC$ ,  $FAB$  are equal: therefore the angle  $HCA$  is equal to the angle  $FBA$ , that is to the angle  $HBL$ . The angle  $LHB$  is equal to the angle  $AHC$ , by I. 15. Therefore the angle  $HLB$  is equal to the angle  $HAC$ , by I. 32. Thus the angle  $HLB$  is a right angle.

142. Since  $EB$  is equal to  $EF$ , the angle  $EBF$  is equal to the angle  $EFB$ ; that is the angle  $OBL$  is equal to the angle  $CFL$ . Therefore the angle  $FCL$  is equal to the angle  $LOB$ , by I. 32 and Exercise 141. Thus the angle  $ECO$  is equal to the angle  $BOL$ , and therefore to the angle  $EOC$ , by I. 15. Therefore  $EO$  is equal to  $EC$  by I. 6; and therefore also equal to  $EA$ .

Thus the angles  $EOC$  and  $EOA$  are together equal to the two angles  $ECO$  and  $EAO$ ; that is the angle  $AOC$  is equal to the two angles  $ACO$  and  $CAO$ : therefore the angle  $AOC$  is a right angle by I. 32.

143. In II. 11 it is shewn that the rectangle  $AB, BH$  is equal to the square on  $AH$ . Therefore the rectangle  $AH, HB$  together with the square on  $BH$  is equal to the square on  $AH$ . Thus the rectangle  $AH, HB$  is equal to the difference of the squares on  $AH$  and  $BH$ , that is to the rectangle contained by the sum and the difference of  $AH$  and  $BH$ : See the note on page 269 of the *Euclid* respecting II. 5 and II. 6.

## II. 12 to 14.

144. Let  $ABC$  be a triangle having the sides  $AB, AC$  equal. From  $B$  draw  $BD$  perpendicular to  $CA$ . By II. 13 we know that the square on  $AC$  is less than the squares on  $AB$  and  $BC$  by twice the rectangle  $AC, CD$ . But the squares on  $AC$  and  $BC$  are equal, by hypothesis; therefore the square on  $BC$  is equal to twice the rectangle  $AC, CD$ .

145. See page 293 of the *Euclid*.

146. The squares on  $CD$  and  $AC$  are equal to twice the squares on  $AB$  and  $CB$ , by Exercise 145. And  $AC$  is equal to  $AB$ . Therefore the square on  $CD$  is equal to the square on  $AB$  together with twice the square on  $CB$ .

147. Let  $ABCD$  be the parallelogram; join  $AC$  and  $BD$  intersecting at  $O$ ; then  $AC$  and  $BD$  are bisected at  $O$ , by Exercise 78.

The squares on  $AB$  and  $BC$  are equal to twice the squares on  $AO$  and  $OB$ , by Exercise 145; and similarly the squares on  $AD$  and  $DC$  are equal to twice the squares on  $AO$  and  $OD$ . Therefore the squares on the sides of the parallelogram are equal to four times the square on  $AO$  and four times the square on  $BO$ , that is to the square on  $AC$  and the square on  $BD$ .

148. Let  $ABC$  be the triangle, and  $O$  the middle point of  $AB$ . By Exercise 145 the squares on  $AC$  and  $BC$  are equal to twice the squares on  $AO$  and  $OC$ . Now  $AO$  is given, and  $OC$  is of constant length: therefore the sum of the squares on  $AC$  and  $BC$  is invariable.

149. Let  $ABCD$  be the quadrilateral; let  $E, F, G, H$  be the middle points of  $AB, BC, CD, DA$  respectively: the squares on  $AC$  and  $BD$  will be equal to twice the sum of the squares on  $EG$  and  $FH$ .

We know that  $EFGH$  is a parallelogram, that  $EH$  is half  $BD$ , and  $HG$  is half  $AC$ : see Exercises 107 and 109. Therefore the squares on  $AC$  and  $BD$  are equal to four times the squares on  $EH$  and  $HG$ , that is to twice the sum of the squares on the sides of the parallelogram  $EFGH$ , that is twice the sum of the squares on the diagonals  $EG$  and  $FH$ , by Exercise 147.

150. Let  $ABCD$  be the parallelogram; let the diagonals  $AC$  and  $BD$  intersect at  $O$ . Let a circle be described from  $O$  as centre, and let  $P$  be any point on the circumference of this circle. Then, by Exercise 145, the sum of the squares on  $PA$  and  $PC$  is equal to twice the squares on  $PO$  and  $AO$ ; and this remains constant at whatever point of the circumference  $P$  may be. Similarly the sum of the squares on  $PB$  and  $PD$  remains constant. Therefore the sum of the squares on  $PA$ ,  $PB$ ,  $PC$ ,  $PD$  remains constant.

151. Let  $ABCD$  be the quadrilateral; let  $E$  be the middle point of  $AC$ , and  $F$  the middle point of  $BD$ .

The squares on  $AB$  and  $BC$  are equal to twice the squares on  $AE$  and  $BE$  by Exercise 145; and similarly the squares on  $AD$  and  $DC$  are equal to twice the squares on  $AE$  and  $DE$ ; therefore the squares on the sides of the quadrilateral are equal to four times the square on  $AE$  together with twice the squares on  $BE$  and  $DE$ , that is to the square on  $AC$  together with twice the squares on  $BE$  and  $DE$ .

But the squares on  $BE$  and  $DE$  are equal to twice the squares on  $BF$  and  $EF$  by Exercise 145; and therefore twice the squares on  $BE$  and  $DE$  are equal to four-times the square on  $BF$  together with four times the square on  $EF$ ; that is to the square on  $BD$  together with four times the square on  $EF$ .

Therefore the squares on the sides of the quadrilateral are equal to the squares on the diagonals together with four times the square on  $EF$ .

152. Let  $O$  be the centre of the circle. By Exercise 145 the squares on  $EC$  and  $ED$  are equal to twice the squares on  $OE$  and  $OC$ , that is to twice the squares on  $AO$  and  $OC$ . Now one of the two straight lines  $AC$  and  $AD$  is the sum of  $AO$  and  $OC$ , and the other is the difference; hence the squares on  $AC$  and  $AD$  are equal to twice the squares on  $AO$  and  $OC$ : see the note on page 269 of the *Euclid* respecting II. 7; or Exercise 138.

Therefore the squares on  $EC$  and  $ED$  are together equal to the squares on  $AC$  and  $AD$ .

153. Let  $O$  be the middle point of  $AD$ . By Exercise 145 the squares on  $AB$  and  $BD$  are equal to twice the squares on  $BO$  and  $OD$ ; and the squares on  $AC$  and  $CD$  are equal to twice the squares on  $CO$  and  $OD$ . But the squares on  $AB$  and  $BD$  are equal to the squares on  $AC$  and  $CD$  by supposition; hence twice the squares on  $BO$  and  $OD$  are equal to twice the squares on  $CO$  and  $OD$ . Therefore the square on  $BO$  is equal to the square on  $CO$ , so that  $BO$  is equal to  $CO$ .

154. Let  $AB$  be the base of an isosceles triangle  $ABC$ ; from  $C$  draw  $CO$  perpendicular to  $AB$ : take any point  $D$  in the base, then will the square on  $AC$  exceed the square on  $DC$  by the rectangle  $AD$ ,  $DB$ .

Suppose  $D$  to fall between  $A$  and  $O$ . The square on  $AC$  is equal to the squares on  $AO$ ,  $OC$  by I. 47; and similarly the square on  $DC$  is equal to the squares on  $DO$ ,  $OC$ . Hence the square on  $AC$  exceeds the square on  $DC$  by the difference of the squares on  $AO$  and  $DO$ , ~~but~~ is by the rectangle contained by the sum and the difference of  $AO$  and  $DO$ : see page 269 of the *Euclid*. Now  $BD$  is equal to the sum of  $AO$  and  $DO$ ; and  $AD$  is the difference of  $AO$  and  $DO$ . Therefore the square on  $AC$  exceeds the square on  $DC$  by the rectangle  $AD$ ,  $DB$ . The process is similar if  $D$  falls between  $B$  and  $O$ .



155. From  $D$  draw  $DF$  perpendicular to  $AB$  produced, and from  $E$  draw  $EG$  perpendicular to  $AC$  produced. The angles  $DBF$  and  $CBA$  are together equal to a right angle, by I. 13. The angles  $ACB$  and  $CBA$  are together equal to a right angle, by I. 32. Therefore the angle  $DBF$  is equal to the angle  $ACB$ .

In the two triangles  $DBF$  and  $BCA$  the sides  $DB$  and  $BC$  are equal; the angle  $DBF$  is equal to the angle  $BCA$ , and the right angles  $DFB$  and  $BAC$  are equal: therefore the triangles are equal in all respects. Similarly it may be shewn that the triangles  $CEG$  and  $BCA$  are equal in all respects.

The square on  $DA$  is equal to the squares on  $DB$ ,  $BA$ , and twice the rectangle  $BA$ ,  $BF$ , by II. 12; that is the square on  $DA$  is equal to the squares on  $BC$ ,  $BA$  and twice the rectangle  $BA$ ,  $AC$ : therefore the squares on  $DA$  and  $AC$  are equal to the squares on  $BC$ ,  $BA$ ,  $AC$  and twice the rectangle  $BA$ ,  $AC$ . Similarly it may be shewn that the squares on  $EA$  and  $AB$  are equal to the squares on  $BC$ ,  $BA$ ,  $AC$  and twice the rectangle  $BA$ ,  $AC$ . Therefore the squares on  $DA$  and  $AC$  are equal to the squares on  $EA$  and  $AB$ .

156. By II. 13 the square on  $BD$  is less than the squares on  $BA$  and  $AD$  by twice the rectangle  $AB$ ,  $AE$ ; and by the same Proposition the square on  $BD$  is less than the squares on  $BA$  and  $AD$  by twice the rectangle  $AC$ ,  $AD$ ; therefore the rectangle  $AB$ ,  $AE$  is equal to the rectangle  $AC$ ,  $AD$ .

157. Let  $ABC$  be an equilateral triangle; suppose  $AB$  produced to  $D$  so that the rectangle  $AD$ ,  $DB$  is equal to the square on  $CB$ : then will the square on  $DC$  be equal to twice the square on  $CB$ .

From  $C$  draw  $CE$  perpendicular to  $AB$ ; then the triangles  $CEA$  and  $CEB$  will be equal in all respects by I. 26. The square on  $DC$  is equal to the squares on  $CB$ ,  $BD$  and twice the rectangle  $DB$ ,  $BE$ , by II. 12; that is to the squares on  $CB$ ,  $BD$  and the rectangle  $DB$ ,  $BA$ ; that is to the square on  $CB$  and the rectangle  $AD$ ,  $DB$ , by II. 3; that is to twice the square on  $CB$ , by supposition.

158. Let  $ABC$  be a triangle, having the angle  $C$  a right angle; from  $C$  draw  $CD$  perpendicular to  $AB$ : then will the square on  $CD$  be equal to the rectangle  $AD$ ,  $DB$ .

Bisect  $AB$  at  $E$ . Then one of the two straight lines  $AD$  and  $DB$  is equal to the sum of  $AE$  and  $ED$ , and the other is equal to their difference. Hence the rectangle  $AD$ ,  $DB$  is equal to the difference of the squares on  $AE$  and  $ED$ : see the *Euclid*, page 269. But  $EC$  is equal to  $AE$ , by Exercise 59. Therefore the rectangle  $AD$ ,  $DB$  is equal to the difference of the squares on  $EC$  and  $ED$ ; that is to the square on  $CD$ , by I. 47.

159. Use the same diagram as in Exercise 158. Then the square on  $BC$  is equal to the squares on  $CD$  and  $DB$ , by I. 47; that is to the rectangle  $AD$ ,  $DB$  and the square on  $DB$ , by Exercise 158; that is to the rectangle  $AB$ ,  $DB$ , by II. 3.

Similarly the square on  $AC$  is equal to the rectangle  $BA$ ,  $DA$ .

160. By II. 13 the square on  $AC$  together with twice the rectangle  $AB$ ,  $BF$  is equal to the squares on  $AB$ ,  $BC$ ; and the square on  $AB$  together with twice the rectangle  $AC$ ,  $CE$  is equal to the squares on  $AC$ ,  $BC$ . Hence the squares on  $AC$ ,  $AB$ , together with twice the rectangle  $AB$ ,  $BF$  and twice

the rectangle  $AC, CE$  are equal to the square on  $AB$ , the square on  $AC$ , and twice the square on  $BC$ . Therefore twice the rectangles  $AB, BF$  and  $AC, CE$  are equal to twice the square on  $BC$ ; and therefore the rectangles  $AB, BF$  and  $AC, CE$  are equal to the square on  $BC$ .

161. Let  $AB$  be the straight line which is to be divided. Let  $LM$  be a side of the given square. Bisect  $AB$  at  $C$ . From  $L$  draw a straight line  $LO$  at right angles to  $LM$ ; with centre  $M$  and radius equal to  $AC$  describe a circle cutting  $LO$  at  $N$ . From  $CB$  cut off  $CD$  equal to  $LN$ . Then will the rectangle  $AD, DB$  be equal to the square on  $LM$ .

The rectangle  $AD, DB$  is equal to the difference of the squares on  $AC$  and  $CD$ , by II. 5; that is to the difference of the squares on  $MN$  and  $LN$ , by construction; that is to the square on  $LM$ , by I. 47.

### III. 1 to 15.

162. Let  $A$  be the centre of the circle which is to be described,  $B$  the centre of the given circle. Join  $AB$ ; through  $B$  draw a straight line at right angles to  $AB$ , cutting the given circle at  $L$  and  $M$ : then will  $AL$  be the radius of the circle required.

In the two triangles  $ABL$  and  $ABM$ , the side  $AB$  is common;  $BL$  is equal to  $BM$ ; and the right angles  $ABL$  and  $ABM$  are equal: therefore  $AL$  is equal to  $AM$ , by I. 4. Therefore a circle described from the centre  $A$  with the radius  $AL$  will pass through  $M$ , and so will cut the given circle at the extremities of a diameter.

163. Each of the straight lines passes through the centre of the circle, as may be shewn in the manner of III. 1: thus the straight lines intersect at a fixed point.

164. Let the circles cut each other at  $B$  and  $E$ . Through  $B$  draw any straight line meeting one circle again at  $A$  and the other again at  $C$ . Through  $E$  draw a straight line parallel to  $ABC$ , meeting the first circle again at  $D$  and the second circle at  $F$ . Then shall  $AC$  and  $DF$  be equal.

Find  $P$  the centre of the circle  $ABED$ , and  $Q$  the centre of the circle  $BCFE$ . From  $P$  draw  $PK$  perpendicular to  $AB$ ; and produce  $KP$  to meet  $DE$  at  $M$ : then since  $DE$  is parallel to  $AB$  the angles at  $M$  are right angles. Again, from  $Q$  draw  $QL$  perpendicular to  $BC$ , and produce  $LQ$  to meet  $EF$  at  $N$ : then the angles at  $N$  are right angles. Thus  $KLNM$  is a right-angled parallelogram, and therefore  $KL$  is equal to  $MN$ .

$KB$  is half of  $AB$  by III. 3, and  $BL$  is half of  $BC$ : therefore  $KL$  is half of  $AC$ . Similarly  $MN$  is half of  $DF$ . But  $KL$  is equal to  $MN$ : therefore  $AC$  is equal to  $DF$ .

165. Suppose that  $D$  and  $F$  are on the circumference of the circle with centre  $A$ ; and that  $E$  and  $G$  are on the circumference of the circle with centre  $B$ . From  $A$  draw  $AL$  perpendicular to  $FC$ , and  $AP$  perpendicular to  $DC$ ; from  $B$  draw  $BM$  perpendicular to  $CG$  and  $BQ$  perpendicular to  $CE$ . From  $B$  draw  $BX$  perpendicular to  $AL$ , so that  $BX$  is parallel to  $FG$ . Again from  $B$  draw a perpendicular  $BY$  on  $PA$  produced, so that  $BY$  is

parallel to  $DE$ . Then the angle  $BA'Y$  will be equal to the angle  $BAX$ , since, by supposition,  $FG$  and  $DE$  are equally inclined to  $AB$ .

Now by I. 26 it will follow that  $BX$  is equal to  $BY$ ; also, as in Exercise 164, it may be shewn that  $BX$  is equal to half of  $FG$ , and  $BY$  is equal to half of  $DE$ . Therefore  $FG$  is equal to  $DE$ .

If the perpendiculars from  $B$  instead of meeting  $AL$  and  $PA$  produced, meet  $LA$  produced and  $AP$  the process is substantially unchanged.

166. Let  $A$  and  $B$  denote the centres of the two circles. From the process given in the solution of Exercise 165 it follows that the straight line drawn through the point of intersection of the two circles is always *less* than twice  $AB$  except when it is parallel to  $AB$ , and then it is equal to twice  $AB$ . Therefore the greatest possible straight line is obtained by drawing a straight line through one of the points of intersection of the circles parallel to the straight line joining the centres.

167. Let  $C$  denote the centre of the circle,  $A$  the point in the diameter,  $P$  and  $Q$  the extremities of the chord parallel to this diameter. From  $C$  draw  $CD$  perpendicular to  $PQ$ ; then  $PQ$  is bisected at  $D$ , by III. 3.

By Exercise 145 the squares on  $AP$  and  $AQ$  are equal to twice the squares on  $AD$  and  $PD$ ; that is to twice the squares on  $AC$ ,  $CD$ , and  $PD$ ; by I. 47; that is to twice the squares on  $AC$  and  $CP$ .

Now one of the segments of the diameter is the sum of  $AC$  and the radius, and the other is the difference of  $AC$  and the radius. Therefore the sum of the squares on the segments is equal to twice the square on  $AC$  and twice the square on the radius; that is to twice the squares on  $AC$  and  $CP$ ; that is to the squares on  $AP$  and  $AQ$ . See Exercise 138.

168. Let  $O$  be the middle point of  $AB$ ; then the sum of the squares on  $AP$  and  $BP$  is equal to twice the square on  $OP$  together with twice the square on  $AO$ . Hence we require  $OP$  to be the least possible. Join  $O$  with the centre of the circle, and let the joining line cut the circumference at  $Q$ ; then  $Q$  is the required point: for  $OQ$  is less than any other straight line drawn from  $O$  to the circumference, by III. 8.

169. Let  $ABC$  be a circle, and let  $F$  be its centre; let  $DBF$  be another circle, and let  $G$  be its centre; let the second circle fall within the first, and let them touch at  $B$ . Let  $AC$  and  $DE$  be parallel diameters,  $A$  and  $D$  being towards the same parts. Then  $B$ ,  $D$ , and  $A$  will be in a straight line.

Join  $BD$  and  $AD$ . If they are not in a straight line let  $BD$  produced meet  $AC$  at  $H$ . Join  $FG$ ; then  $FG$  produced will pass through  $B$ , by III. 11. The angle  $GDB$  is equal to the angle  $FHB$ , by I. 29. Therefore the angle  $FHB$  is equal to the angle  $FBH$ . Therefore  $FH$  is equal to  $FB$ . Therefore  $FH$  is equal to  $FA$  which is absurd. Therefore  $B$ ,  $D$ , and  $A$  cannot lie otherwise than in a straight line.

If the circles touch *externally* the process will be similar; but then the extremities of the diameters must be taken towards opposite parts.

170. Let  $C$  be the centre of the larger circle,  $A$  the centre of the other. Let  $FAE$  be a chord of the larger circle at right angles to  $AC$ ; and let  $H$  be one of the points where it cuts the smaller circle. Let  $BHD$  be another chord of the larger circle, and let it be at right angles to  $FAE$ . Of the two  $EH$  and  $FH$  let  $FH$  be the less; and of the two  $BH$  and  $DH$  let  $DH$  be the less. Then will  $EH$  be equal to  $BH$ , and  $FH$  equal to  $DH$ .

From  $C$  draw  $CK$  perpendicular to  $BD$ ; then  $ACKH$  will be a square. The chords  $FE$  and  $BD$  are equally distant from  $C$ , and are therefore equal, by III. 14. Therefore  $AE$  and  $BK$ , the halves of these chords, are equal. But  $AH$  is equal to  $HK$ ; therefore  $EH$  is equal to  $BH$ ; and therefore  $FH$  is equal to  $DH$ .

171. Let  $A$  be the given point,  $C$  the centre of the circle. Join  $CA$  and through  $A$  draw a chord  $BAD$  at right angles to  $AC$ : this shall be the shortest chord through  $A$ .

For draw any other chord  $EAF$  through  $A$ ; and from  $C$  draw the perpendicular  $CG$  on  $EAF$ . Then since  $CGA$  is a right angle  $CAG$  is less than a right angle, by I. 82. Therefore  $CG$  is less than  $CA$ , by I. 19. Therefore  $EAF$  is nearer to the centre than  $BAD$ ; and therefore  $EAF$  is greater than  $BAD$ , by III. 15.

172. From  $O$  draw the radius  $OB$  parallel to  $PN$ , so that  $OB$  and  $PN$  are on opposite sides of the diameter on which  $ON$  lies. Join  $PB$ .

The angle  $NPB$  is equal to the angle  $OBP$ , by I. 29; the angle  $OBP$  is equal to the angle  $OPB$  by I. 5: therefore the angle  $NPB$  is equal to the angle  $OPB$ . Thus the angle  $OPN$  is bisected by the straight line  $PB$ ; so that so long as  $P$  is on the same side of the fixed diameter the straight line bisecting the angle always passes through the fixed point  $B$ .

Produce  $BO$  to meet the circumference again at  $B'$ . Then if  $P$  is on the other side of the fixed diameter, the straight line bisecting the angle will pass through  $B'$ .

173. Take  $O$  the centre of the circle  $DBCE$ ; join  $OD$  and  $OE$ . Take  $P$  the centre of the circle on which  $A$  and  $B$  lie, and  $Q$  the centre of the circle on which  $A$  and  $C$  lie.

The angle  $PAB$  is equal to the angle  $PBA$ , and therefore equal to the angle  $OBD$ , and therefore equal to the angle  $ODB$ , by I. 5 and I. 15. Therefore  $PA$  is parallel to  $DO$  by I. 27. Similarly  $QA$  is parallel to  $OE$ . But  $PA$  and  $QA$  form one straight line by III. 12. Therefore  $OD$  and  $OE$  form one straight line, which is a diameter of the circle  $DBCE$ , and is parallel to  $PQ$ .

174. Let  $ABCD$  be the quadrilateral figure. Let the circles described on  $AB$  and  $BC$  as diameters intersect at  $E$ . Then the straight line which bisects  $EB$  at right angles will pass through the centres of the circles, by III. 1, so that it will bisect  $AB$  and  $BC$ . Therefore this straight line will be parallel to  $AE$  and to  $EC$ , by Exercise 106. Hence  $AE$  and  $EC$  are in one straight line; and the common chord  $EB$  is at right angles to  $AC$ . Similarly the common chord of the circles described on  $CD$  and  $DA$  as diameters is also at right angles to  $AC$ ; and therefore the two common chords are parallel.

In like manner the common chord of the circles described on  $AD$  and  $AB$  as diameters is parallel to the common chord of the circles described on  $CB$  and  $CD$  as diameters.

175. Let  $C$  be the centre of the given circle,  $A$  the given point in the given straight line. On this straight line take  $AB$  equal to the radius of the given circle; join  $BC$ ; at  $C$  draw the radius  $CD$ , making the angle  $BCD$  equal to the angle  $CBA$ , and on the same side of  $CB$ . Produce  $DC$  to meet the given straight line at  $O$ . Then  $O$  will be the required centre.

For the angle  $OCB$  is equal to the angle  $OBC$ , by construction; therefore  $OC$  is equal to  $OB$ , by I. 6. And  $BA$  is equal to  $CD$ , by construction; therefore  $OD$  is equal to  $OA$ ; and the circle described from the centre  $O$  with the radius  $OA$  will pass through  $D$ , and will touch the given circle at  $D$ .

### III. 16 to 19.

176. In the diagram of III. 17 let  $FD$  be produced to meet the outer circle at  $H$ ; join  $EH$  cutting the inner circle at  $K$ ; join  $AK$ : then  $AK$  will touch the circle  $BCD$ .

This may be shewn precisely as in III. 17. Also  $DH$  is equal to  $DF$  by III. 3; therefore  $AK$  will be equal to  $AB$ .

177. Let  $C$  be the centre of the given circle. From  $C$  draw a straight line perpendicular to the given straight line meeting it at  $A$ , and intersecting the circumference at  $B$ . Through  $B$  draw a straight line at right angles to  $CB$ ; this touches the given circle by III. 16 Corollary; and it is parallel to the given straight line by I. 28.

178. Let  $C$  be the centre of the given circle. Draw the radius  $CA$  parallel to the given straight line. Through  $A$  draw a straight line at right angles to  $CA$ ; this touches the given circle by III. 16 Corollary; and it is perpendicular to the given straight line by I. 29.

179. Let  $C$  be the centre of the circle,  $A$  the end of the diameter. Draw  $AD$  at right angles to  $CA$ , and make it of the given length. Join  $CD$  cutting the circumference at  $B$ ; from  $B$  draw a straight line at right angles to  $CB$  and let it meet  $CA$  produced at  $E$ . Then  $EB$  will be the required tangent.

For in the two triangles  $CAD$ ,  $CBE$  the angle  $C$  is common, the right angles  $CAD$  and  $CBE$  are equal, and the sides  $CA$  and  $CB$  are equal; hence  $AD$  is equal to  $BE$ , by I. 26. And  $BE$  touches the circle at  $B$ , by III. 16 Corollary.

180. Let  $O$  be the centre of the two circles. Let  $ABC$  be a chord of the outer circle, touching the inner circle at  $B$ ; let  $DEF$  be another chord of the outer circle, touching the inner circle at  $E$ : then will  $AC$  be equal to  $DF$ .

Join  $OB$  and  $OE$ . The angles at  $B$  and  $E$  are right angles, by III. 18. Thus  $AC$  and  $DF$  are equally distant from  $O$ : therefore  $AC$  is equal to  $DF$ , by III. 14.

181. Let  $O$  be the centre of the given circle,  $P$  the given point. In the circle place a chord  $AB$  equal to the given straight line. From  $O$  draw a perpendicular  $OC$  on  $AB$ . With centre  $O$  and radius  $OC$  describe a circle, and from  $P$  draw a straight line touching this circle at  $E$  and meeting the given circle at the points  $D$  and  $F$ . Then  $PDEF$  shall be the straight line required.

For by Exercise 180 the straight lines  $AB$  and  $DF$  are equal.

182. Let  $MCN$  be the diameter; let tangents be drawn at  $M$  and  $N$ ; let  $AB$  be the portion of another tangent which is intercepted between the tangents at  $M$  and  $N$ : then  $ACB$  shall be a right angle.

Let  $AB$  touch the circle at  $E$ . Join  $AC$  and  $BC$ .

In the triangles  $CAE$  and  $CAM$  the sides  $AE$  and  $AM$  are equal, by Exercise 176; the sides  $EC$  and  $MC$  are equal; and the right angles  $AEC$  and  $AMC$  are equal: therefore the angles  $ACE$  and  $ACM$  are equal, by I. 4. Similarly the angles  $BCE$  and  $BCN$  are equal. Therefore the angle  $ACB$  is equal to the two angles  $ACM$  and  $BCN$ . Therefore the angle  $ACB$  is a right angle, by I. 13.

183. Let  $AB$  be the given straight line,  $C$  the centre of the given circle. Draw a straight line  $DE$  parallel to  $AB$ , and at a distance from it equal to the radius of the required circle which is given. With centre  $C$  describe a circle having its radius equal to the sum of the radii of the given circle and the required circle; let this circle cut  $DE$  at the points  $F$  and  $G$ : then either  $F$  or  $G$  may be taken as the centre of the required circle.

For the distance of  $F$  from  $AB$  is equal to the radius of the required circle, and therefore the circle described from  $F$  as centre with the given radius will touch  $AB$ . And the distance of  $F$  from  $C$  is equal to the sum of the radii of the given circle and the required circle; and therefore the circle described from  $F$  as centre will touch the given circle. Similarly  $G$  may be taken as the centre of the required circle.

If the line  $AB$  lies without the given circle,  $DE$  must be drawn on the same side of  $AB$  as  $C$  is.

184. Let  $O$  be the centre of the given circle. Let a circle described from  $C$  as centre touch the given straight line at  $A$ , and touch externally the given circle at  $B$ . From  $O$  draw a straight line parallel to  $CA$  to meet the circumference of the given circle at  $D$ , so that  $OD$  and  $CA$  may be on opposite sides of  $OC$ . Then  $D$  will be a fixed point; and  $D, B, A$  will lie in one straight line.

For  $OD$  and  $CA$  are parallel, by construction; and the angles at  $A$  are right angles, by III. 18: therefore if  $DO$  be produced to meet the given straight line the angles at the point of intersection will be right angles. Thus  $OD$  is a fixed straight line, and  $D$  is a fixed point.

Join  $AB, BD$ . The angle  $BOD$  is equal to the angle  $BCA$  by I. 29. Therefore the angles  $ODB$  and  $OBD$  together are equal to the angles  $CAB$  and  $CBA$  by I. 32. But the angle  $OBD$  is equal to the angle  $ODB$ , and the angle  $CBA$  is equal to the angle  $CAB$ , by I. 5. Therefore the angle  $OBD$  is equal to the angle  $CBA$ . Therefore  $DB$  and  $BA$  are in the same straight line.

185. See the *Euclid*, page 295.

186. Let  $ABC, DEF$  be the circles. It is required to draw a straight line touching the circle  $ABC$ , and so that the part of it intercepted by  $DEF$  may be equal to a given straight line.

In the circle  $DEF$  place the straight line  $DE$  equal to the given straight line. Find  $G$  the centre of this circle, and with  $G$  as a centre describe a circle touching  $DE$ . Draw a straight line  $AFH$  to touch the latter circle and to touch  $ABC$ , cutting the circle  $DEF$  at  $F$  and  $H$ : this shall be the straight line required.

For since  $DE$  and  $FH$  are equally distant from the centre of the circle  $DEF$ , they are equal: therefore  $FH$  is of the required length.

187. Let  $A$  and  $B$  be the centres of the two circles. In the circle having its centre at  $A$  place a chord  $PQ$  of the length which is to be

intercepted by this circle from the required straight line. In the circle having its centre at  $B$  place a chord  $RS$  of the length which is to be intercepted by this circle from the required straight line. With  $A$  and  $B$  as centres describe circles touching these chords respectively. Draw a straight line to touch the two circles thus described: this will be the straight line required.

For let this straight line meet the given circle which has  $A$  for centre at the points  $K$  and  $L$ , and the given circle which has  $B$  for centre at the points  $M$  and  $N$ . Then  $KL$  and  $PQ$  are equally distant from  $A$ , and therefore they are equal by III. 14. Similarly  $MN$  and  $RS$  are equal.

188. Let  $ABCD$  be a quadrilateral, the sides of which touch a circle: then will  $AB$  and  $CD$  together be equal to  $BC$  and  $DA$  together.

Let  $AB$  touch the circle at  $P$ , let  $BC$  touch it at  $Q$ , let  $CD$  touch it at  $R$ , and let  $DA$  touch it at  $S$ . Then  $AP$  is equal to  $AS$ ,  $BP$  to  $BQ$ ,  $CQ$  to  $CR$ , and  $DR$  to  $DS$ , by Exercise 176. Therefore the sum of  $AP$ ,  $PB$ ,  $CR$ ,  $RD$  is equal to the sum of  $AS$ ,  $BQ$ ,  $CQ$ ,  $DS$ ; that is  $AB$  and  $CD$  together are equal to  $BC$  and  $DA$  together.

189. Let  $ABCD$  be a parallelogram which is described about a circle. Then  $AB$  and  $CD$  together are equal to  $BC$  and  $DA$  together, by the preceding Exercise. But  $AB$  is equal to  $CD$ , and  $BC$  is equal to  $DA$ , by I. 34. Hence  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  are all equal, so that the parallelogram is a rhombus.

190. If possible suppose that  $DE$  does not touch the circle but cuts it. Draw  $DF$  to touch the circle at  $F$  and produce  $DF$  to meet  $AE$  produced at  $G$ .

Then  $DF$  is equal to  $DB$ , and  $GF$  is equal to  $GC$ , by Exercise 176. Therefore  $DG$  is equal to the sum of  $DB$  and  $GC$ . But, by hypothesis,  $DE$  is equal to the sum of  $DB$  and  $EC$ ; add  $EG$  to both; thus  $DE$  and  $EG$  are equal to the sum of  $DB$ ,  $EC$ , and  $EG$ ; that is to the sum of  $DB$  and  $GC$ . Therefore  $DE$  and  $EG$  are equal to  $DG$ ; that is two sides of a triangle are equal to the third, which is impossible by I. 20. Therefore  $DE$  does not cut the circle. In the same way it may be shewn that  $DE$  does not fall without the circle. Therefore  $DE$  touches the circle.

191. Take the diagram of Exercise 188. Let  $O$  be the centre of the circle. Draw  $OA$ ,  $OB$ ,  $OC$ ,  $OD$ ,  $OP$ ,  $OQ$ ,  $OR$ ,  $OS$ .

In the triangles  $OAS$ ,  $OAP$  the side  $OA$  is common; the sides  $OS$  and  $OP$  are equal; the sides  $AS$  and  $AP$  are equal by Exercise 176. Therefore the angles  $AOS$  and  $AOP$  are equal, by I. 8. Similarly the angles  $BOP$  and  $BOQ$  are equal; also the angles  $COQ$  and  $COR$  are equal; and the angles  $DOR$  and  $DOS$  are equal. Therefore the angles  $AOP$ ,  $BOP$ ,  $COR$ ,  $DOR$  are together equal to the angles  $AOS$ ,  $BOQ$ ,  $COQ$ ,  $DOS$ ; therefore each set of four angles are together equal to two right angles, by I. 15. Corollary 2.

192. Let  $CA$  and  $CB$  be the two radii which are at right angles. Let a straight line touch the circle, and meet  $CA$  produced at  $P$ , and  $CB$  produced at  $Q$ . From  $P$  draw  $PM$  to touch the circle, and from  $Q$  draw  $QN$  to touch the circle; then  $PM$  will be parallel to  $QN$ .

It may be shewn as in Exercise 191 that the angle  $MPQ$  is double the angle  $CPQ$ , and that the angle  $NQP$  is double the angle  $CQP$ . Therefore the two angles  $MPQ$  and  $NQP$  are double the angles  $CPQ$  and  $CQP$ . But the angles  $CPQ$  and  $CQP$  are together equal to a right angle, by I. 32:

therefore the angles  $MPQ$  and  $NQP$  are together equal to two right angles. Therefore  $PM$  and  $QN$  are parallel, by I. 28.

193. Let a straight line touch at  $C$  a circle which has its centre at  $A$ , and touch at  $D$  a circle which has its centre at  $B$ . Let  $DC$  and  $BA$  produced meet at  $O$ . Let the straight line  $OAB$  meet the first circle at  $E$  and the second circle at  $F$ , where  $E$  and  $F$  are taken on the same sides of  $A$  and  $B$  respectively. Then  $CE$  will be parallel to  $DF$ .

The angles  $OCA$  and  $ODB$  are right angles, by III. 18; therefore the angles  $CAO$  and  $DBO$  are equal, by I. 32. Therefore the two angles  $ACE$  and  $AEC$  are together equal to the two  $BDF$  and  $DFB$ , by I. 32. But the angles  $ACE$  and  $AEC$  are equal, and the angles  $BFD$  and  $BDF$  are equal, by I. 5. Therefore the angle  $AEC$  is equal to the angle  $BFD$ . Therefore  $CE$  is parallel to  $DF$ , by I. 28.

194. Let  $ABCD$  be a quadrilateral. Let a circle touch  $AB$  at  $E$ , touch  $BC$  at  $F$ , and touch  $DA$  at  $L$ ; let another circle touch  $BC$  at  $G$ , touch  $CD$  at  $H$ , and touch  $DA$  at  $K$ . Also let the two circles touch each other at  $O$ . Let the straight line  $MON$  be drawn touching both circles at  $O$ , and meeting  $AD$  at  $M$ , and  $BC$  at  $N$ . Then  $AD$  and  $BC$  together will exceed  $AB$  and  $DC$  together, by twice  $MN$ .

By Exercise 176 we see that  $AB$  and  $CD$  together are equal to  $AL$ ,  $DK$ ,  $CG$ , and  $FB$ : thus the excess of  $AD$  and  $BC$  together above  $AB$  and  $DC$  together is equal to  $LK$  and  $FG$  together. But  $LM$  and  $MK$  are each equal to  $MO$ , and  $FN$  and  $NG$  are each equal to  $NO$ . Therefore  $LK$  and  $FG$  together are equal to twice  $MN$ .

195. Let  $D$  be the point where the circle and the semicircle touch. Then  $D$ ,  $O$ ,  $C$  are in one straight line, III. 11. Let  $F$  be the point at which the circle touches  $AB$ , and produce  $FO$  to meet at  $E$  the tangent to the semicircle parallel to  $AB$ . Then will  $CQ$  be equal to  $OE$ .

For  $FE$  is equal to the radius of the semicircle, because  $E$  is in the tangent to the semicircle which is parallel to  $AB$ . Thus  $FE$  is equal to  $CD$ . Also  $FO$  is equal to  $OD$ . Therefore  $OC$  is equal to  $OE$ .

196. From any point  $A$  without a circle having its centre at  $O$  let two straight lines  $AB$ ,  $AC$  be drawn touching the circle. The angle  $BAC$  will be double the angle  $OBC$ .

The angles  $ABC$  and  $OBC$  together make up a right angle; so also do the angles  $ACB$  and  $BAO$ ; therefore the angle  $OBC$  is equal to  $BAO$ . But the angle  $BAC$  is double the angle  $BAO$ ; therefore the angle  $BAC$  is double the angle  $OBC$ .

197. Let  $AB$  be the diameter of a circle,  $C$  the centre. At any point  $D$  on the circumference let a tangent be drawn, meeting at  $E$  the tangent at  $A$ , and meeting at  $F$  the tangent at  $B$ . The area of the figure  $ABFE$  will be half that of a rectangle contained by  $AB$  and  $EF$ .

The triangle  $EAC$  is equal to the triangle  $EDC$ , and the triangle  $FDC$  is equal to the triangle  $FBC$ , by I. 8 and Exercise 176. Hence the figure  $ABFE$  is equal to twice the triangle  $ECF$ ; and is therefore equal to a rectangle contained by  $EF$  and  $DC$ , and is therefore equal to half a rectangle contained by  $EF$  and  $AB$ .



198. Let  $ABCD$  be the quadrilateral, having  $BC$  parallel to  $AD$ . Let  $K$  be the centre of a circle which touches  $AB$  at  $E$ ,  $BC$  at  $F$ ,  $CD$  at  $G$ , and  $DA$  at  $H$ . Through  $K$  draw a straight line  $LKM$  parallel to  $AD$ , and terminated by  $AB$  and  $DC$ . Then  $LM$  will be equal to one-fourth of the perimeter of  $ABCD$ .

It may be shewn that  $AB$  is bisected at  $L$ , and  $DC$  bisected at  $M$ : see Exercise 36.

The angle  $EBK$  is equal to the angle  $FBK$ , by Exercise 176. The angle  $FBK$  is equal to the angle  $LKB$ , by I. 29. Therefore the angle  $LBK$  is equal to the angle  $LKB$ . Therefore  $LK$  is equal to  $LB$ , by I. 6. Thus  $LK$  is half  $AB$ . Similarly  $MK$  is half  $DC$ . Thus  $LM$  is half the sum of  $AB$  and  $DC$ ; that is  $LM$  is one-fourth of the perimeter of  $ABCD$ , by Exercise 188.

199. Let  $A$  be the fixed point in the fixed straight line. Let a circle which touches the fixed straight line at  $A$  cut the fixed parallel straight line at  $B$  and  $C$ . Draw a tangent to this circle at  $B$ , and let it meet the fixed straight line through  $A$  at  $F$ ; and draw from  $A$  the perpendicular  $AE$  on this tangent; also draw from  $A$  the perpendicular  $AD$  on  $BC$ .

The angle  $FBA$  is equal to the angle  $FAB$ , by Exercise 176; and is therefore equal to the angle  $ABD$ , by I. 29. Hence in the two triangles  $ABE$  and  $ABD$  the side  $AB$  is common; the angle  $ABE$  is equal to the angle  $ABD$ ; and the right angles  $AEB$  and  $ADB$  are equal: therefore  $AE$  is equal to  $AD$ , by I. 26. Hence the straight line  $FB$  will touch at  $E$  the circle described from  $A$  as a centre with a radius equal to the fixed length  $AD$ .

Similarly the tangent at  $C$  to the circle which passes through  $A$ ,  $B$ ,  $C$  will touch the same fixed circle.

200. Let  $A$  and  $B$  be the two given points. Let  $C$  be a point on the convex circumference of the given circle such that the tangent to the circle at  $C$  is equally inclined to  $AC$  and  $CB$ . The sum of  $AC$  and  $BC$  shall be less than the sum of the straight lines drawn from  $A$  and  $B$  to any other point on the circumference.

For take any other point on the circumference, as  $P$ ; join  $PA$  and  $PB$ . Let  $PA$  cut at  $E$  the tangent at  $C$ , and join  $EB$ . Then the sum of  $PE$  and  $PB$  is greater than  $EB$ , by I. 20. Therefore the sum of  $PA$  and  $PB$  is greater than the sum of  $EA$  and  $EB$ . But the sum of  $EA$  and  $EB$  is greater than the sum of  $CA$  and  $CB$ , by page 306 of the *Euclid*. Therefore the sum of  $PA$  and  $PB$  is greater than the sum of  $CA$  and  $CB$ .

201. Join  $CD$ . The angle  $CDA$  is equal to the angle  $CAD$ , by I. 5. The angles  $CDB$  and  $BDE$  are together equal to a right angle, by III. 18. The angles  $CBA$  and  $CAB$  are together equal to a right angle, by I. 32. Therefore the angle  $BDE$  is equal to the angle  $CBA$ . Also the angle  $DBE$  is equal to the angle  $CBA$ , by I. 15. Therefore the angles  $BDE$  and  $DBE$  are equal; and the triangle  $DBE$  is isosceles, by I. 6.

202. Let  $O$  be the centre of the circle. Produce  $OC$  to a point  $F$  such that  $CF$  is equal to  $CO$ : then by comparing the triangles  $PCO$  and  $PCF$  it may be shewn that  $PF$  is equal to  $PO$ , and also  $OF$  is equal to  $OP$ , so that  $POF$  is an equilateral triangle. Therefore the angle  $POG$  is two-thirds of a right angle. Therefore the angle  $EPA$  is one-third of a right angle, by

- I. 32. Therefore the angle  $PEA$  is two-thirds of a right angle, by I. 32. Therefore the angle  $CED$  is two-thirds of a right angle, by I. 15.

The angle  $COA$  is equal to the angles  $OBC$  and  $OCB$ , by I. 32, that is to twice the angle  $OBC$ , by I. 5: therefore the angle  $OBC$  is one-third of a right angle. The angle  $BAD$  is a right angle. Therefore the angle  $ADB$  is two-thirds of a right angle. Thus each of the angles  $CDE$  and  $CED$  is equal to two-thirds of a right angle; and therefore the angle  $ECD$  is also two-thirds of a right angle, by I. 32. Thus the triangle  $DEC$  is equiangular, and therefore equilateral.

### III. 20 to 22.

203. The four angles  $ABD$ ,  $BDC$ ,  $DCA$ ,  $CAB$  are equal to four right angles. The angle  $BAC$  is fixed, and the angle  $BDC$  is constant, by III. 21: therefore the sum of the angles  $ABD$  and  $ACD$  is constant.

204. The angle  $RAB$  is half the sum of the angles  $PAB$  and  $QAB$ ; and the angle  $RBA$  is half the sum of the angles  $PBA$  and  $QBA$ . Thus the angle  $R$  together with half the sum of the angles  $PAB$ ,  $QAB$ ,  $PBA$ ,  $QBA$  is equal to two right angles. Again, half the sum of the angles  $Q$ ,  $QAB$ ,  $QBA$  is equal to a right angle; and half the sum of the angles  $P$ ,  $PAB$ ,  $PBA$  is equal to a right angle. Thus half the sum of the angles  $P$  and  $Q$  together with half the sum of the angles  $PAB$ ,  $QAB$ ,  $PBA$ ,  $QBA$  is equal to the angle  $R$  together with half the sum of the angles  $PAB$ ,  $QAB$ ,  $PBA$ ,  $QBA$ . Therefore the angle  $R$  is equal to half the sum of the angles  $P$  and  $Q$ , and is therefore constant by III. 21.

205. In the quadrilateral  $QCPD$  the angles  $QCP$ ,  $CPD$ ,  $PDQ$  are all constant by III. 21. The four angles of a quadrilateral are together equal to four right angles. Therefore the angle  $CQD$  is constant.

206. The angles at  $Q$  and  $R$  are constant by III. 21; therefore the third angle of the triangle formed on  $QR$  as a base by  $QA$  and  $RB$  produced is constant, by I. 32.

207. The angles  $ODC$  and  $ADC$  are together equal to two right angles by I. 13; but the angle  $ODC$  is equal to the angle  $ABC$ ; therefore the angles  $ABC$  and  $ADC$  are together equal to two right angles. Therefore a circle can be described round  $ABCD$ : see the *Euclid*, page 276.

208. Let  $ABCD$  be the quadrilateral inscribed in a circle. Let  $AB$  produced through  $B$ , and  $DC$  produced through  $C$ , meet at  $O$ .

The angles  $BAD$  and  $BCD$  are together equal to two right angles, by III. 22. The angles  $BCD$  and  $BCO$  are together equal to two right angles by I. 13. Therefore the angles  $BAD$  and  $BCD$  are together equal to the two angles  $BCD$  and  $BCO$ . Therefore the angle  $BCO$  is equal to the angle  $BAC$ . Similarly the angle  $CBO$  is equal to the angle  $ADO$ .

209. Let  $ABCD$  be a parallelogram inscribed in a circle. By III. 22 the angles  $A$  and  $C$  are together equal to two right angles; by I. 34 the angles  $A$  and  $C$  are equal: therefore each of the angles  $A$  and  $C$  is a right angle.

Similarly each of the angles  $B$  and  $D$  is a right angle.

210. Let  $ABC$  be a triangle inscribed in a circle. Let  $D$ ,  $E$ ,  $F$  be points in the segments exterior to the triangle cut off by  $BC$ ,  $CA$ ,  $AB$  respectively. The angles  $CAB$  and  $CDB$  are together equal to two right angles, by III. 22; so also are the angles  $CBA$  and  $CEA$ , and the angles  $ACB$  and  $AFB$ . Therefore the angles in the exterior segments together with the angles of the triangle are together equal to six right angles; therefore the angles in the exterior segments are together equal to four right angles, by I. 32.

211. Take the diagram of III. 22. Let  $ABCD$  be a quadrilateral inscribed in a circle. The angle in the segment exterior to  $AB$  and the angle  $ACB$  are together equal to two right angles, by III. 22. So also are the angle in the segment exterior to  $BC$  and the angle  $CAB$ . Also the angle in the segment exterior to  $CD$  and the angle  $CBD$ . Also the angle in the segment exterior to  $DA$  and the angle  $DBA$ . Therefore the angles in the four exterior segments together with the angles  $ACB$ ,  $CAB$ ,  $CBD$ ,  $DBA$  are equal to eight right angles. But the angles  $ACB$ ,  $CAB$ ,  $CBD$ ,  $DBA$  together make up the angles of the triangle  $ABC$ , and are therefore equal to two right angles. Hence the angles in the four exterior segments are equal to six right angles.

212. Let  $ACB$  be a triangle inscribed in a circle such that the angle  $ACB$  is one-third of two right angles. In the segment cut off by  $AB$  on the opposite side of  $C$  take any point  $D$  on the circumference. Then the angles  $ACB$  and  $ADB$  are together equal to two right angles, by III. 22. But the angle  $ACB$  is one-third of two right angles, by construction. Therefore the angle  $ADB$  is two-thirds of two right angles. Thus the angle in the segment  $ADB$  is double the angle in the segment  $ACB$ .

213. Let  $ACB$  be a triangle inscribed in a circle such that the angle  $ACB$  is one-sixth of two right angles. In the segment cut off by  $AB$  on the opposite side of  $C$  take any point  $D$  on the circumference. Then the angles  $ACB$  and  $ADB$  are together equal to two right angles, by III. 22. But the angle  $ACB$  is one-sixth of two right angles by construction. Therefore the angle  $ADB$  is five-sixths of two right angles. Thus the angle in the segment  $ADB$  is five times the angle in the segment  $ACB$ .

214. Let  $ABCD$  be a quadrilateral. Let a side  $AB$  be produced to  $E$ , and suppose that the angle  $EBC$  is equal to the angle  $ADC$ .

The angles  $ABC$  and  $EBC$  are equal to two right angles by I. 13. But the angle  $EBC$  is equal to the angle  $ADC$ , by supposition. Therefore the angles  $ABC$  and  $ADC$  are together equal to two right angles. Therefore a circle can be circumscribed round  $ABCD$ : see the *Euclid*, page 276. Then the angles  $ADB$  and  $ACB$  will be equal by III. 21. In like manner any other side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

215. Let  $ABCDEF$  be a hexagon inscribed in a circle; suppose that  $AB$  is parallel to  $DE$ , and  $BC$  parallel to  $EF$ : then will  $CD$  be parallel to  $FA$ .

Join  $AD$ . The angle  $ABC$  is equal to the angle  $FED$  by Exercise 35. The angles  $FED$  and  $FAD$  are together equal to two right angles by III. 22; so also are the angles  $ABC$  and  $ADC$ . Therefore the angles  $FED$  and

$FAD$  are together equal to the angles  $ABC$  and  $ADC$ . Therefore the angle  $FAD$  is equal to the angle  $ADC$ . Therefore  $FA$  is parallel to  $DC$ , by I. 27.

216. Let the straight lines which respectively bisect the angles  $APC$  and  $AQC$  meet at  $H$ : the angle  $PHQ$  will be a right angle.

The angle  $PCQ$  and the angle  $BAD$  are together equal to two right angles, by I. 15 and III. 22. Therefore the angles  $PQC$  and  $QPC$  are together equal to the angle at  $A$ . Now the angle  $CQH$  and half the angles at  $A$  and  $B$  are equal to a right angle by I. 32; so also the angle  $CPH$  together with half the angles at  $A$  and  $D$  are equal to a right angle. Thus the angles  $CQH$ ,  $CPH$ , together with the angle at  $A$ , and half the angles at  $B$  and  $D$  are equal to two right angles; therefore the angles  $CQH$ ,  $CPH$ , and  $A$  are together equal to a right angle, by III. 22. Therefore, by what is shewn above, the angles  $HQP$  and  $HPQ$  are together equal to a right angle. Therefore the angle  $PHQ$  is a right angle, by I. 32.

217. Let  $ABCD$  be a quadrilateral inscribed in a circle. Let a straight line  $EFGH$  be drawn meeting the circumference at  $E$  and  $H$ , the side  $AD$  at  $F$ , and the side  $BC$  at  $G$ ; and suppose the angle  $AFG$  to be equal to the angle  $BGF$ : then will  $EH$  also make equal angles with  $AB$  and  $DC$ .

Since the angles  $AFG$  and  $BGF$  are equal the arcs  $ED$  and  $ABH$  are together equal to the arcs  $HC$  and  $EAB$ : see the *Euclid*, page 294. Therefore the difference between the arcs  $HC$  and  $ED$  is equal to the difference between the arcs  $BH$  and  $EA$ . Therefore  $HE$  makes equal angles with  $AB$  and  $DC$ : see the *Euclid*, page 294.

218. Let  $EFGH$  be a quadrilateral which can be inscribed in a circle, so that the angles  $E$  and  $G$  are together equal to two right angles. Suppose also that a circle can be inscribed in this quadrilateral touching the sides  $EF$ ,  $FG$ ,  $GH$ ,  $HE$  at  $A$ ,  $B$ ,  $C$ ,  $D$  respectively. Then will the straight lines  $BD$  and  $AC$  be at right angles to each other.

Let  $O$  be the centre of the circle  $ABCD$ . The angle between  $AC$  and  $BD$  is measured by half the sum of the arcs  $DA$  and  $BC$ ; see the *Euclid*, page 294, and is therefore equal to half the sum of the angles  $DOA$  and  $BOC$ . Now the angles  $ODE$  and  $OAE$  are right angles; therefore  $DOA$  and  $DEA$  are together equal to two right angles, by I. 32. But the angles  $DEA$  and  $BGC$  are together equal to two right angles, by supposition. Therefore the angle  $DOA$  is equal to the angle  $BGC$ . Similarly the angle  $BOC$  is equal to the angle  $AED$ . Therefore the angle between  $AC$  and  $BD$  is equal to half the sum of the angles  $AED$  and  $BGC$ , that is to a right angle.

### III. 23 to 30.

219. Let  $AC$  and  $BG$  be chords of two equal arcs in a circle; let  $AB$  and  $CG$  join the extremities of these chords towards the same parts: then  $AB$  and  $CG$  shall be parallel.

The angle  $ABC$  is equal to the angle  $BCG$  by III. 27: therefore  $CG$  is parallel to  $AB$  by I. 27.

220. Let  $AB$  and  $CG$  be parallel chords in a circle; let them be joined towards the same parts by  $AC$  and  $BG$ , and towards opposite parts by  $AG$  and  $BC$ : then  $AC$  will be equal to  $BG$ , and  $AG$  to  $BC$ .

Because  $AB$  is parallel to  $CG$  the angle  $ABC$  is equal to the angle  $BCG$ , by I. 29: therefore the arc  $AC$  is equal to the arc  $BG$ , by III. 26: therefore the straight line  $AC$  is equal to the straight line  $BG$ , by III. 29.

Again, it has been shewn that the arc  $AC$  is equal to the arc  $BG$ ; to each add the arc  $CG$ : therefore the arc  $ACG$  is equal to the arc  $BGC$ : therefore the straight line  $AG$  is equal to the straight line  $BC$ , by III. 29.

221. Suppose  $C$  and  $D$  on opposite sides of  $A$ , and  $C$  and  $E$  on opposite sides of  $B$ . Then the angle  $DAE$  is the sum of the angles  $ACB$  and  $AEB$ , which are invariable by III. 21, and is therefore invariable. Therefore the arc  $DE$  is invariable, by III. 26.

But  $C$  and  $D$  may be on the same side of  $A$ , and  $C$  and  $E$  on the same side of  $B$ . Thus the angle  $ECD$  is equal to the angle  $ACB$ , and is therefore invariable, by III. 21. Also the angle  $AEB$  is invariable, by III. 21. Therefore the angle  $EAD$  which is the difference of  $ECD$  and  $AEB$ , by I. 32, is also invariable. Therefore the arc  $DE$  is invariable, by III. 26.

222. Let the straight line which bisects the angles meet the arc  $CB$  at a point  $P$ : then will  $EP$  be equal to  $BP$ .

The angles  $ECP$  and  $EBP$  are together equal to two right angles, by III. 22; the angles  $ECP$  and  $DCP$  are equal to two right angles, by I. 13: therefore the angle  $EBP$  is equal to the angle  $DCP$ , that is to half the angle  $DCB$ . Again, the angle  $BEP$  is equal to the angle  $BCP$ , by III. 21, that is to half the angle  $DCB$ . Thus the angles  $EBP$  and  $BEP$  are equal; therefore  $EP$  and  $BP$  are equal, by I. 6.

223. Let  $ABCD$  be a quadrilateral figure inscribed in a circle; produce  $CD$  to any point  $E$ ; let the straight lines which bisect the angles  $ABC$  and  $ADE$  meet at  $F$ : then  $F$  shall be on the circumference of the circle.

Let  $AD$  and  $BF$  intersect at  $H$ . The angle  $ADE$  is equal to the angle  $ABC$ , by III. 22: therefore the angle  $HDF$  is equal to the angle  $HBA$ . The angle  $DHF$  is equal to the angle  $BHA$ , by I. 15. Therefore the angle  $DFH$  is equal to the angle  $BAH$ , by I. 32. Therefore  $F$  is on the circumference of the circle. See the *Euclid*, page 276.

If  $BF$  meet  $AD$  produced at  $H$ , the proof will be similar, using the converse of III. 22 instead of that of III. 21.

224. Let  $C$  be the centre of the given circle. From  $D$  as centre with radius equal to  $CD$  describe a circle cutting  $AB$  produced at  $F$ . Join  $FD$  and produce it to meet the given circle again at  $E$ . Then will the arc  $AE$  be three times the arc  $BD$ .

For the angle  $ACE$  is equal to the two angles  $CEF$  and  $CFE$ , by I. 32; therefore the angle  $ACE$  is equal to the two angles  $CDE$  and  $CFD$ , by I. 5; therefore the angle  $ACE$  is equal to the angle  $DCF$  and twice the angle  $DFC$ , by I. 32; therefore the angle  $ACE$  is equal to three times the angle  $DCF$ , by I. 5. Hence the arc  $AE$  is three times the arc  $BD$ , by III. 26.

225. The points  $P$  and  $Q$  will always lie on a fixed segment of a circle described on  $AB$ : see the *Euclid*, page 276. Also as the angle  $AQB$  is given, and the angle  $C$  is given, the angle  $QBP$  is invariable. Hence the straight line  $PQ$  is always of the same length.

226. Let  $AB$  be the common chord of the two equal circles. Through  $A$  draw a straight line meeting one circle at  $C$ , and the other at  $D$ ; draw  $BC$  and  $BD$ : these straight lines shall be equal.

For the angle  $ACB$  is equal to the angle  $ADB$ , by III. 27 and III. 28: therefore  $BC$  is equal to  $BD$  by I. 6.

227. In the two triangles  $BOP$ ,  $BOQ$  the side  $OB$  is common; the angle  $BOP$  is equal to the angle  $BOQ$ , by supposition; and the right angles  $BPO$  and  $BQO$  are equal: therefore  $BP$  is equal to  $BQ$ , by I. 26.

Now  $AB$  is equal to  $BC$ , by III. 26 and III. 29. Therefore the square on  $AB$  is equal to the square on  $BC$ . Therefore the squares on  $AP$  and  $PB$  are equal to the squares on  $CQ$  and  $QB$ . But  $PB$  has been shewn equal to  $QB$ . Therefore the square on  $AP$  is equal to the square on  $CQ$ . Therefore  $AP$  is equal to  $CQ$ .

228. Suppose  $AB$  to lie between  $AL$  and  $AM$ , and  $AM$  nearer to the centre than  $AL$ . From  $B$  draw  $BP$  perpendicular to  $AM$  and  $BQ$  perpendicular to  $AL$  produced. Then, as in Exercise 227, we have  $PM$  equal to  $QL$ . Therefore the sum of  $AL$  and  $AM$  is equal to twice  $AP$ ; and this is a fixed quantity, because the side  $AB$  is fixed, and so are the two angles  $PAB$  and  $APB$ .

Next let  $AB$  not lie between  $AL$  and  $AM$ , and suppose  $AM$  nearer to the centre than  $AL$ . From  $B$  draw  $BP$  perpendicular to  $AM$  and  $BQ$  perpendicular to  $LA$  produced. Now  $LB$  and  $MB$  are equal by Exercise 222; and thus we find as in Exercise 227 that  $LQ$  and  $PM$  are equal. Hence the difference of  $AM$  and  $AL$  will be found to be equal to twice  $AP$ , and this as before is a fixed quantity.

229. Let  $FE$  and  $OA$  be produced to meet at  $H$ . Bisect  $OF$  at  $K$ , and join  $EK$ .

The triangles  $EFK$  and  $EOK$  are equal in all respects by I. 8; so that  $EKO$  is a right angle: therefore  $KE$  and  $OH$  are parallel by I. 28. Therefore the angle  $FEK$  is equal to the angle  $EOH$ , and the angle  $KEO$  is equal to the angle  $EOH$ , by I. 29. Therefore the angle  $EOH$  is equal to the angle  $EOH$ .

The angle  $GOB$  is equal to the two angles  $OGH$  and  $OHG$ , by I. 32; therefore the angle  $GOB$  is equal to the two angles  $OEG$  and  $OHG$ , by I. 5; therefore the angle  $GOB$  is equal to the angle  $EOH$  and twice the angle  $OHG$ ; therefore the angle  $GOB$  is equal to three times the angle  $EOH$ . Hence the arc  $BG$  is three times the arc  $AE$ , by III. 26.

230. Let  $AB$  be the given base, and  $ABC$  one of the triangles. The point  $C$  will be situated on the arc of a certain fixed segment of a circle described on  $AB$  as a base: see the *Euclid*, page 276. Let the angle at  $C$  be bisected by a straight line, and let this straight line be produced to cut at  $D$  the circumference of the circle of which the segment is part. Then since the angles  $ACD$  and  $BCD$  are equal the arcs  $AD$  and  $BD$  are equal; so that  $D$  is the middle point of the arc  $ADB$ . Hence  $D$  is a fixed point, and the straight lines which bisect the vertical angles all pass through this fixed point.

231. Let the two circles touch at  $A$ . Let  $K$  be the centre of the smaller circle, and  $L$  the centre of the larger circle. Let  $BDC$  be a chord of the larger circle, touching the smaller circle at  $D$ . Then  $BD$  and  $DC$  will subtend equal angles at  $A$ .

Produce  $AD$  to cut the larger circle at  $F$ . Join  $KD$  and  $LF$ . Then  $A$ ,  $K$ ,  $L$  are in one straight line, by III. 11. The angle  $KAD$  is equal to the

angle  $KDA$ , and the angle  $LAD$  is equal to the angle  $LFA$ , by I. 5: therefore the angle  $KDA$  is equal to the angle  $LFA$ : therefore  $LF$  is parallel to  $KD$ , by I. 28. But  $KD$  is at right angles to  $BC$ , by III. 18; therefore  $LF$  is at right angles to  $BC$ , by I. 29. Therefore  $LF$  bisects  $BC$ , by III. 3. Therefore the arc  $BF$  is equal to the arc  $CF$ . Therefore the angle  $BAF$  is equal to the angle  $CAF$ , by III. 27.

## III. 31.

232. Let  $AB$  be the given hypotenuse, and  $ABC$  one of the triangles having the angle at  $C$  a right angle. Let  $D$  be the middle point of  $AB$ . Then  $DC$  is equal to  $DA$  by Exercise 59, so that  $C$  is on the circumference of a circle having  $D$  as centre, and  $DA$  as radius, and therefore having  $AB$  as diameter.

233. Let  $ABC$  be a triangle having the sides  $AC$  and  $BC$  equal. Let  $D$  be the middle point of  $AB$ . Then the triangles  $ADC$  and  $BDC$  are equal in all respects, by I. 8; so that the angles  $ADC$  and  $BDC$  are right angles. Hence the circle described on  $AC$  as diameter will pass through  $D$ , by Exercise 232; and also the circle described on  $BC$  as diameter will pass through  $D$ . Thus the circles intersect at  $D$ .

234. Let  $ABCD$  be a rectangle inscribed in a circle. Since the angle  $ADC$  is a right angle  $AC$  is a diameter of the circle. Let  $O$  be the centre of the circle. The rectangle is double of the triangle  $ADC$  by I. 34. The triangle  $DAC$  has a fixed base, namely the diameter of the circle; but its height is always less than  $OD$  except when  $DOA$  is a right angle. Therefore the greatest rectangle that can be inscribed in a circle is one which has its diameters at right angles to each other; this by I. 4 has its sides all equal, and is therefore a square.

235. Suppose  $C$  and  $E$  on opposite sides of  $AB$ , and  $C$  and  $A$  on opposite sides of  $FE$ .  $CD$  is equal to half  $AB$ , by Exercise 59. Therefore a circle described from the centre  $D$  with radius  $CD$  will pass through  $B, E, A, F$ . The angle  $ECB$  is half the angle  $EDB$ , by III. 20; that is half a right angle, and is therefore half the angle  $ACB$ , by III. 31. Thus  $EC$  bisects the angle  $ACB$ .

Similarly the angle  $FCA$  is half the angle  $FDA$ , that is half a right angle. Thus  $FC$  bisects the angle between  $CA$  and  $BC$  produced.

236. Let  $O$  be the centre of the circle. Let  $E$  and  $C$  be on the same side of  $AB$ .  $EF$  is parallel to  $BC$ ; therefore the angle  $OEB$  is equal to the angle  $EBC$ , by I. 29. But the angle  $OEB$  is equal to the angle  $OBE$ , by I. 5. Therefore the angle  $OBE$  is equal to the angle  $EBC$ . Thus  $EB$  bisects the angle  $OBC$ .

Similarly if  $CB$  be produced to  $D$  it may be shewn that the angle  $OBD$  is bisected by  $BF$ .

237. A circle described on  $AC$  as diameter will pass through  $D$  and  $E$ , by Exercise 232. Then  $ACE$  and  $ADE$  will be angles in the same segment, and therefore equal by III. 21.

238. The angles  $ABC$  and  $ABD$  are both right angles, by III. 31. Therefore  $BC$  and  $BD$  are in one straight line by I. 14. That is the straight line  $CD$  passes through  $B$ .

239. Draw through  $A$  any chord  $AC$  of the larger circle, and let it intersect the smaller circle at  $B$ . Then  $OBA$  is a right angle by III. 31. Therefore  $OB$  bisects  $AC$ , by III. 3.

240. Let  $A, B, C$  be three given points in a given straight line; and let it be required to describe a circle which shall touch the straight line at  $B$ , and have the tangents drawn to the circle from  $A$  and  $C$  parallel.

On  $AC$  as diameter describe a semicircle. From  $B$  draw a straight line at right angles to the given straight line, and let it meet the semicircle at  $O$ . From  $O$  as centre with radius equal to  $OB$  describe a circle: this will be the circle required.

For this circle touches  $ABC$  at  $B$ , by III. 16 Cor. Draw  $AD$  and  $CE$  to touch this circle. Thus  $AD$  is equal to  $AB$ , by Exercise 176: therefore the triangles  $OAB$  and  $OAD$  are equal in all respects, so that the angle  $OAD$  is equal to the angle  $OAB$ , and therefore the angle  $DAB$  is double the angle  $OAB$ . Similarly the angle  $ECB$  is double the angle  $OCB$ . Therefore the angles  $DAB$  and  $ECB$  are together double the angles  $OAB$  and  $OCB$ . But the angle  $AOC$  is a right angle, by III. 31; therefore the angles  $OAC$  and  $OCA$  are together equal to a right angle, by I. 32. Therefore the angles  $DAB$  and  $ECB$  are together equal to two right angles. Therefore  $AD$  and  $CE$  are parallel, by I. 28.

241. Let  $A$  and  $B$  be the given points in the given straight line; so that the tangents from  $A$  and  $B$  to the required circle are to be parallel; let  $C$  denote the centre of the required circle. Then as in Exercise 182, or as in Exercise 240, we see that the angle  $ACB$  must be a right angle. Therefore  $C$  must lie on the circumference of a circle described on  $AB$  as a diameter, see the *Euclid*, page 276.

Again, since the circle is to have a given radius and to touch  $AB$  the centre  $C$  must lie on a straight line which is parallel to  $AB$ , and at a distance from  $AB$  equal to the given radius.

Therefore describe a circle on  $AB$  as a diameter, and draw a straight line parallel to  $AB$  and at a distance from  $AB$  equal to the given radius: then the intersection of the straight line and the circle will determine the centre of the required circle which can therefore be drawn.

242. Let  $ABC$  be a triangle; from  $B$  draw  $BE$  perpendicular to  $AC$ , produced if necessary; from  $C$  draw  $CF$  perpendicular to  $AB$ , produced if necessary. Let  $D$  be the middle point of  $BC$ .

Then  $DE$  and  $DF$  are each equal to  $DC$ , by Exercise 59. Therefore a circle described from  $D$  as centre, with radius equal to  $DC$ , will pass through  $B, F$ , and  $E$ . Therefore the perpendicular from  $D$  on  $EF$  will bisect  $EF$ , by III. 3.

243. The angle  $ABD$  is a right angle, and the angle  $BCD$  is obtuse by III. 31. The square on  $AD$  is equal to the sum of the squares on  $AB$  and  $BD$ , by I. 47. Therefore the square on  $AD$  is equal to the squares on  $AB, BC, CD$  and twice the rectangle  $BC, CE$ , by II. 12.

244. Let  $C$  be the middle point of  $AM$ . Let  $PM$  and  $QR$  intersect at  $D$ . Produce  $DQ$  to any point  $E$ .

Then  $PQMR$  is a rectangle, by III. 31; therefore  $DQ$  equals  $DP$ . The angle  $EQC$  is equal to the angles  $EQA$  and  $AQC$ ; and therefore equal to the angles  $PQD$  and  $AQC$ ; and therefore equal to the angles  $QPD$  and  $AQC$ ;



and therefore equal to the angles  $APM$  and  $PAM$ ; and therefore equal to a right angle, by I. 32. Therefore  $QR$  touches the circle  $AQM$  at  $Q$ .

Similarly it may be shewn that  $QR$  touches the circle  $BRM$  at  $R$ .

245. A circle described on  $BC$  as diameter will pass through  $F$  and  $D$ ; also a circle described on  $FD$  as diameter will pass through  $E$  and  $G$ ; see the *Euclid*, page 276. By the aid of the latter circle we have the angle  $FEG$  equal to the angle  $FDG$ : by the aid of the former circle we have the angle  $FBC$  equal to the angle  $FDG$ . Therefore the angle  $FEG$  is equal to the angle  $FBC$ . Therefore  $EG$  is parallel to  $BC$ , by I. 28.

246. Take the case in which the chords  $BC$  and  $AC$  are *produced through C* to meet the other circumference. Let  $HC$  be a diameter of the circle  $ABC$ , and let it be produced to cut  $ED$  at  $K$ .

The angle  $CED$  is equal to the angle  $CAB$ , by III. 21; and therefore equal to the angle  $CHB$ , by III. 21. The angle  $ECK$  is equal to the angle  $HCB$ , by I. 15. Therefore the angle  $EKC$  is equal to the angle  $HBC$ , by I. 32. Therefore the angle  $EKC$  is a right angle, by III. 31.

The figure becomes modified in other cases, but the demonstration remains substantially unchanged.

247. Let  $ABC$  be a triangle having a right angle at  $C$ . Let squares be described on  $BC$ ,  $CA$ ,  $AB$  outside the triangle; and let  $D$ ,  $E$ ,  $F$  respectively be the intersections of the diagonals in these squares.

Then the angle  $AFB$  is a right angle, and therefore a circle would go round  $ACBF$ : see in the *Euclid* the note on III. 22. The angle  $FCA$  will be equal to the angle  $FBA$ , by III. 21. Therefore the angle  $FCA$  is half a right angle. Also the angle  $ACE$  is half a right angle. Therefore the angle  $FCE$  is a right angle. So also is  $FCD$ . Then use I. 14.

248. Draw a straight line at right angles to  $CA$  at the point  $A$ ; let this meet the circumference of the given circle at  $B$ : then  $B$  will be the point required.

For a circle described on  $CB$  as diameter will go through  $A$ , and will touch the given circle at  $B$ . Let  $P$  be any other point on the circumference of the given circle. Join  $CP$  cutting at  $Q$  the circle just described; and join  $AQ$ .

Then the angle  $CBA$  is equal to the angle  $CQA$ , by III. 21. The angle  $CQA$  is greater than the angle  $CPA$  by I. 16. Therefore the angle  $CBA$  is greater than the angle  $CPA$ .

249. Let  $FG$  produced meet  $AB$  at  $H$ . The angles  $FDG$  and  $FEG$  are right angles, by III. 31: therefore a circle would go round  $FDGE$ , by page 276 of the *Euclid*. Therefore the angle  $DFG$  is equal to the angle  $DEG$ ; and therefore equal to the angle  $DBA$ , by III. 21. Hence in the triangles  $AFH$  and  $ABD$  the angle at  $A$  is common; and the angle  $AFH$  is equal to the angle  $ABD$ ; therefore the angle  $AHF$  is equal to the angle  $ADB$ , and is therefore a right angle, by III. 31.

250. Let two equal circles touch externally at  $B$ . Let  $AB$  be a diameter of one circle, and  $BC$  be a diameter of the other. Draw  $BD$ , a chord of the first circle; and  $BE$  at right angles to  $BD$  a chord of the second circle. The straight line  $DE$  shall be parallel to  $AC$  and equal to the distance between the centres of the given circles.

Join  $AD$  and  $CE$ . The angle  $ADB$  is a right angle, by III. 31; the angle  $DBE$  is a right angle by supposition; therefore the angles  $ADB$  and  $DBE$  are equal: therefore  $AD$  is parallel to  $BE$ , by I. 27. Therefore the angle  $BAD$  is equal to the angle  $CBE$ , by I. 29. The angle  $ADB$  is equal to the angle  $BEC$ , by III. 31. Also  $AB$  is equal to  $BC$ . Therefore  $BE$  is equal to  $AD$ , by I. 26. And it was shewn that  $BE$  is parallel to  $AD$ . Therefore  $DE$  is equal and parallel to  $AB$ , by I. 33.

251. Let  $ACBD$  be the rhombus, and  $AB$  the shorter diagonal. On  $AB$  as diameter describe a circle cutting  $AC$  at  $H$ ,  $BC$  at  $K$ ,  $AD$  at  $L$ ,  $BD$  at  $M$ . Join  $AK$  and  $BH$ , intersecting at  $E$ . Join  $AM$  and  $BL$  intersecting at  $F$ . Then  $AEBF$  will be a rhombus, having its angles equal to those of  $DACB$ .

Since  $AC$  is equal to  $BC$  the angle  $CAB$  is equal to the angle  $CBA$ . In the triangles  $BAH$  and  $ABK$  the angles  $HAB$  and  $KBA$  are equal, as we have just shewn; the angles  $AHB$  and  $AKB$  are equal, being right angles by III. 31: therefore the angles  $ABH$  and  $BAK$  are equal, by I. 32. Therefore  $EA$  is equal to  $EB$ , by I. 5.

Also  $AC$  is parallel to  $BD$ , by Exercise 76; therefore the angle  $HAB$  is equal to the angle  $MBA$ ; and the right angle  $AHB$  is equal to the right angle  $BMA$ ; therefore the angle  $ABH$  is equal to the angle  $BAM$ . Therefore  $BE$  is parallel to  $AF$ . Similarly  $AE$  is parallel to  $BF$ . Thus the figure  $AEBF$  is a parallelogram. And since  $EA$  has been shewn equal to  $EB$  the parallelogram is a rhombus.

Again, the angle  $AEB$  is equal to the angle  $HEK$ , by I. 15. The angles  $HEK$  and  $HCK$  are together equal to two right angles, because the angles  $EHC$  and  $EKC$  are right angles. The angles  $HCK$  and  $CAD$  are equal to two right angles, by I. 29. Therefore the angles  $HCK$  and  $CAD$  are together equal to the angles  $HCK$  and  $HEK$ . Therefore the angle  $CAD$  is equal to the angle  $HEK$ , that is to the angle  $AEB$ . This shews that the angles of one rhombus are equal to the angles of the other.

252. Let  $AB$ ,  $CD$  be two chords of a circle; let them meet, produced if necessary, at  $E$  at right angles. The squares on  $AE$ ,  $EB$ ,  $CE$ ,  $ED$  will be together equal to the square on the diameter.

Let  $O$  be the centre of the circle. Draw  $OG$  perpendicular to  $AB$ , and  $OH$  perpendicular to  $CD$ . Then  $AB$  is bisected at  $G$ , and  $CD$  is bisected at  $H$ , by III. 3. The squares on  $AE$  and  $EB$  are together double of the squares on  $AG$  and  $EG$ , and the squares on  $CE$  and  $ED$  are together double of the squares on  $CH$  and  $HE$ ; see the *Euclid*, page 269. Also  $EG$  is equal to  $HO$ , and  $EH$  is equal to  $GO$ , by I. 34. Therefore the squares on  $AE$ ,  $EB$ ,  $CE$ ,  $ED$  are equal to double the squares on  $AG$ ,  $OG$ ,  $CH$ ,  $HO$ ; that is to double the squares on  $OA$  and  $OC$ , by I. 47; that is to four times the square on  $OA$ , that is to the square on the diameter of the circle.

### III. 32 to 34.

253. Produce  $BC$  to meet the circumference again at  $E$ ; and produce  $PD$  to meet the circumference again at  $F$ .

Then  $PD$  is equal to  $FD$  by III. 3. The triangles  $BPD$  and  $BFD$  are equal in all respects, by I. 4; so that the angle  $BPD$  is equal to the angle  $BFD$ ; and therefore equal to the angle  $BEP$ , by III. 21. The angle  $BPA$

is equal to the angle  $BEP$ , by III. 32. Therefore the angle  $BPA$  is equal to the angle  $BPD$ .

254. Let two circles touch each other at the point  $B$ . Let a straight line  $ABC$  meet one circle at the points  $A$  and  $B$ , and the other circle at the points  $B$  and  $C$ . The segments cut off by  $AB$  will be respectively similar to those cut off by  $BC$ .

Draw a straight line  $EBF$  through  $B$ , at right angles to the straight line which joins the centres of the circles; then this straight line touches both circles by III. 16 Cor. The angles in the segments cut off by  $AB$  are equal to the angles  $ABE$  and  $ABF$  respectively, by III. 32; so also are the angles in the segments cut off by  $BC$ . Hence the segments cut off by  $AB$  are respectively similar to those cut off by  $BC$ , by the definition of similar segments.

255. The angle  $DAP$  is equal to the angle  $PQA$ , by III. 32; and therefore equal to the angle  $QAB$ , by I. 29. The angle  $DPA$  is equal to the angle  $QBA$ , by I. 13 and III. 22. Therefore the angle  $PDA$  is equal to the angle  $BQA$ , by I. 32.

256. Suppose  $G$  to fall between  $A$  and  $H$ . The angle  $GBD$  is equal to the angle  $GAB$ , by III. 32; the angle  $GAB$  is equal to the angle  $HDB$ , by III. 21; therefore the angle  $GBD$  is equal to the angle  $HDB$ . Therefore  $HD$  is parallel to  $BG$ , by I. 27.

The figure becomes modified in other cases but the demonstration remains substantially unchanged.

257. Produce  $CA$  to any point  $E$ , and  $DA$  to any point  $F$ . The angle  $CAF$  is equal to the angle  $ABC$ , and the angle  $DAE$  is equal to the angle  $ABD$ , by III. 32. But the angle  $CAF$  is equal to the angle  $DAE$ , by I. 15: therefore the angle  $ABC$  is equal to the angle  $ABD$ . Therefore  $AB$ , produced if necessary, bisects the angle  $CBD$ .

258. Draw  $PE$  the tangent at  $P$ , so that  $PE$  and  $PA$  are on the same side of  $PB$ . The angle  $EPA$  is equal to the angle  $PBA$ , by III. 32. The angle  $PBA$  is equal to the angle  $PCD$ , by I. 13 and III. 22. Therefore the angle  $EPA$  is equal to the angle  $PCD$ . Therefore  $EP$  is parallel to  $CD$ , by I. 27.

259. Let  $A$  be any point in the circumference of a circle. From  $A$  draw the chord  $AB$ , and the tangent  $AC$ . From  $D$ , the middle point of the arc  $AB$ , draw  $DE$  perpendicular to the chord  $AB$ , and  $DF$  perpendicular to the tangent  $AC$ . Then  $DE$  will be equal to  $DF$ .

The angle  $DAF$  is equal to the angle  $DBA$ , by III. 32. The angle  $DBA$  is equal to the angle  $DAB$ , by III. 27. Therefore the angle  $DAF$  is equal to the angle  $DAE$ . Then in the two triangles  $DAF$  and  $DAE$  the side  $AD$  is common; the angle  $DAF$  is equal to the angle  $DAE$ ; and the right angle  $DFA$  is equal to the right angle  $DEA$ : therefore  $DF$  is equal to  $DE$ , by I. 26.

260. The angle  $APN$  is equal to the angle  $AQM$ , by III. 32. The right angle  $ANP$  is equal to the right angle  $AMQ$ . Therefore the angle  $PAN$  is equal to the angle  $QAM$ , by I. 32. Therefore the angle  $NAM$  is equal to the angle  $PAQ$ .

Again, since  $ANP$  and  $AMP$  are right angles a circle would go round  $ANPM$ ; see the *Euclid*, page 276. Therefore the angle  $ANM$  is equal to

the angle  $APQ$ , by III. 21. Therefore the angle  $AMN$  is equal to the angle  $AQP$ , by I. 32.

261. The angle  $RPS$  is a right angle, by III. 31. The angle  $QPS$  is equal to the angle  $PAB$ , by III. 32 and I. 15. But the angle  $PAB$  is equal to the angle  $OSB$ , by I. 32; that is equal to the angle  $PSQ$ . Therefore the angle  $QPS$  is equal to the angle  $QSP$ .

Again, since  $RPS$  is a right angle the angles  $PSR$  and  $PRS$  are together equal to the angle  $RPS$ , by I. 32. And it has been shewn that the angle  $QPS$  is equal to the angle  $QSP$ ; therefore the angle  $QPR$  is equal to the angle  $QRP$ . Therefore by I. 6 we have  $QS$  and  $QR$  each equal to  $QP$ , so that  $QS$  is equal to  $QR$ .

262. Let  $AB$  be the given base, and  $D$  the point in the base on which the perpendicular falls. On  $AB$  describe a segment of a circle containing an angle equal to the given vertical angle. From  $D$  draw a straight line at right angles to  $AB$ , and let it cut the arc of the segment at  $C$ . Then  $ABC$  is the triangle required. For this triangle has the given base, and the given vertical angle; and the perpendicular from the vertex on the given base meets the base at the assigned point.

263. Let  $AB$  be the given base. On  $AB$  describe a segment of a circle containing an angle equal to the given vertical angle. From  $A$  draw a straight line  $AD$  at right angles to  $AB$ , and equal in length to the given altitude. Through  $D$  draw a straight line parallel to  $AB$ , and let it cut the arc of the segment at  $C$  and  $E$ . Then the triangle  $ACB$  will satisfy all the prescribed conditions; and so also will the triangle  $AEB$ .

264. Let  $AB$  be the given base. On  $AB$  describe a segment of a circle containing an angle equal to the given vertical angle. From the middle point of  $AB$  as centre with a radius equal to the given length describe a circle, and let it cut the arc of the segment at  $C$  and  $E$ . Then the triangle  $ACB$  will satisfy all the prescribed conditions, and so also will the triangle  $AEB$ .

265. Let  $AB$  be the given base. On  $AB$  describe a segment of a circle containing an angle equal to the given vertical angle. From the middle point of  $AB$  draw a straight line at right angles to  $AB$ , and let it meet the arc of the segment at  $C$ . Then the triangle  $ABC$  is the greatest triangle that can be constructed under the prescribed conditions.

For the straight line  $DC$  passes through the centre of the circle of which the segment forms part, by III. 1. Hence a straight line through  $C$  parallel to  $AB$  will be a tangent to the circle, by III. 16 Cor. If any other point except  $C$  on the arc of the segment be taken as the vertex of the triangle the vertex will be nearer to  $AB$  than  $C$  is; so that the triangle will have a *less* height than  $ACB$ , and therefore will be less than  $ACB$ .

266. We suppose that  $B$  is nearer to  $A$  than  $C$  is. We may regard  $OB$  as the base of the triangle; then the height of the triangle will always be less than  $OC$  except when the angle  $COB$  is a right angle, and then the height of the triangle is  $OC$ . Hence the triangle  $OBC$  is greatest when the angle  $COB$  is a right angle. Then as the angles  $OBC$  and  $OCB$  are equal each of them will be half a right angle.

Therefore on  $OA$  describe a segment of a circle containing an angle equal to half a right angle, and take the point at which the arc of this segment cuts the given circle for the point  $C$ .

267. Suppose  $A$  and  $B$  the fixed points. Let  $C$  be a point in which the two straight lines meet. Then all the points of intersection which are on the same side of  $AB$  as  $C$  will lie on the arc of a segment of a circle described on  $AB$  with an angle equal to  $ACB$ : see the *Euclid*, page 276. All the straight lines which bisect these angles pass through a fixed point, by Exercise 230.

Similarly a second fixed point is obtained by considering the straight lines bisecting the angles formed at points which lie on the side of  $AB$  opposite to that we have hitherto considered.

268. Let  $AB$  be the side which is opposite to the given angle. On  $AB$  describe a segment containing an angle equal to the given angle, and also a segment containing an angle equal to half the given angle. From  $A$  as centre with radius equal to the sum of the other two sides describe a circle, and let  $D$  be one of the points where it intersects the second described segment. Join  $AD$  and let it cut the first described segment at  $C$ . Then the triangle  $ACB$  will satisfy the prescribed conditions.

The angle  $ACB$  is equal to the sum of the angles  $CDB$  and  $CBD$ , by I. 32; also the angle  $ACB$  is double the angle  $CDB$  by construction: therefore the angle  $CDB$  is equal to the angle  $CBD$ : therefore  $CD$  is equal to  $CB$ , by I. 6.

Hence  $CA$  and  $CB$  together are equal to  $CA$  and  $CD$  together; that is to  $AD$ ; that is to the given quantity. And  $AB$  and the angle  $ACB$  also have the prescribed values.

### III. 35 to 37.

269. Let two circles cut one another at the points  $A$  and  $B$ . From any point  $T$  in  $AB$  produced suppose a tangent  $TP$  drawn to one circle, and a tangent  $TQ$  to the other: then will  $TP$  be equal to  $TQ$ .

For by III. 36 the rectangle  $TA, TB$  is equal to the square on  $TP$ , and also to the square on  $TQ$ ; therefore the square on  $TP$  is equal to the square on  $TQ$ ; therefore  $TP$  is equal to  $TQ$ .

270. Let  $PQ$  denote the common tangent; and let  $AB$  produced meet  $PQ$  at  $T$ . Then  $TP$  is equal to  $TQ$ , by Exercise 269: therefore  $PQ$  is bisected at  $T$ .

271. Since the angles  $ADC$  and  $AEC$  are equal a circle will go round  $AEDC$ . Therefore the rectangle  $BC, BD$  is equal to the rectangle  $BA, BE$ , by III. 36, Corollary.

272. Let  $AB$  be the common chord of the two circles. Through any point  $C$  in  $AB$  draw  $DCE$  a chord of one circle, and  $FCG$  a chord of the other. The rectangle  $AC, CB$  is equal to the rectangle  $DC, CE$ , and also equal to the rectangle  $FC, CG$ , by III. 35; therefore the rectangle  $DC, CE$  is equal to the rectangle  $FC, CG$ . Therefore a circle will go through  $D, F, E, G$ : see the *Euclid*, page 277.

273. Let  $BC$  be the given straight line, and  $C$  the fixed point in it. Let  $A$  be the given centre. Join  $AC$ ; and describe a circle on  $AC$  as diameter. In it place a straight line  $CD$  equal to a side of the given square. With centre  $A$  and radius equal to  $AD$  describe a circle: this will be the circle required.

For let this circle cut  $BC$  at  $E$  and  $F$ . The angle  $ADC$  is a right angle, by III. 31; therefore  $CD$  touches the circle  $DEF$ , by III. 16 Cor.; therefore the rectangle  $CE, CF$  is equal to the square on  $CD$  by III. 36.

274. Let  $K$  be the middle point of  $BC$ . The square on  $GE$  is equal to the rectangle  $GB, GC$ , by III. 36. Thus the square on  $GE$  is equal to the difference of the squares on  $GK$  and  $BK$ ; see the *Euclid*, page 269. Therefore the square on  $GK$  exceeds the square on  $GE$  by the square on  $BK$ . Therefore four times the square on  $GK$  exceeds four times the square on  $GE$  by four times the square on  $BK$ . Therefore four times the square on  $GK$  exceeds the square on  $AE$  by the square on  $BC$ : see Exercise 270. And it may be shewn that  $K$  is the middle point of  $GH$ ; so that four times the square on  $GK$  is the square on  $GH$ . Thus the square on  $GH$  exceeds the square on  $AE$  by the square on  $BC$ .

To shew that  $K$  is the middle point of  $GH$  we may proceed thus. First  $AE$  and  $DF$  will be equal; for each of them will be a side of a right-angled triangle in which the hypotenuse is the distance of the centres of the circles, and a side is equal to the difference of the radii of the circles: see the *Euclid*, page 295. Next the square on  $GK$  is equal to the squares on  $GE$  and  $BK$ ; and the square on  $HK$  is equal to the squares on  $HF$  and  $CK$ ; therefore the square on  $GK$  is equal to the square on  $HK$ : therefore  $GK$  is equal to  $HK$ .

275. Let two of the circles intersect at  $A$  and  $B$ . Let  $T$  be the fixed point such that the tangent  $TP$  to one circle is equal to the tangent  $TQ$  to the other. Then  $A, B$ , and  $T$  shall be on a straight line.

For if not, join  $TA$  and let  $TA$ , produced if necessary, cut one circle at  $L$  and the other at  $M$ . Then by III. 36 the square on  $TP$  is equal to the rectangle  $TA, TL$ ; and the square on  $TQ$  is equal to the rectangle  $TA, TM$ . Therefore the rectangle  $TA, TL$  is equal to the rectangle  $TA, TM$ ; which is impossible. Therefore  $TA$  cannot cut the circles at any other point than  $B$ .

276. Describe a circle on  $BC$  as diameter; and let the straight line  $DE$  when produced cut this circle at  $N$  and  $O$ .

The rectangle  $NE, EO$  is equal to the difference of the squares on  $DN$  and  $DE$ ; see the *Euclid*, page 269. But by III. 35 the square on  $DN$  is equal to the rectangle  $BD, DC$ ; and the rectangle  $NE, EO$  is equal to the rectangle  $AE, EC$ . Therefore the rectangle  $AE, EC$  is equal to the difference of the rectangle  $BD, DC$  and the square on  $DE$ . Therefore the square on  $DE$  is equal to the difference of the rectangles  $BD, DC$  and  $AE, EC$ .

Again, the rectangle  $FN, FO$  is equal to the difference of the squares on  $DF$  and  $DN$ ; see the *Euclid*, page 269. So that the square on  $DF$  is equal to the square on  $DN$  together with the rectangle  $FN, FO$ . But the square on  $DN$  is equal to the rectangle  $DC, DB$ , by III. 35; and the rectangle  $FN, FO$  is equal to the rectangle  $FA, FB$ , by III. 36, Corollary. Therefore the square on  $DF$  is equal to the sum of the rectangles  $BD, DC$  and  $AF, FB$ .

277. Let  $AB$  be the given diameter. In the circle place a straight line  $BP$  equal to a side of the given square. Join  $AP$  and produce it to meet at  $T$  the tangent to the circle at  $B$ . Then  $T$  is the required point.

For the rectangle  $TP$ ,  $TA$  is equal to the square on  $TB$ , by III. 36; therefore the rectangle  $TP$ ,  $PA$  together with the square on  $TP$  is equal to the square on  $TB$ , by II. 3. Therefore the rectangle  $TP$ ,  $PA$  is equal to the difference of the squares on  $TB$  and  $TP$ ; that is to the square on  $BP$ , by III. 31 and I. 47.

#### IV. 1 to 4.

278. Join  $AB$ . The straight lines through  $A$  and  $B$  to touch the circle make with  $AB$  angles which are together less than two right angles, for each of them is less than one right angle. Therefore, by Axiom 12, these straight lines will meet.

279. The angles  $ABC$  and  $ACB$  are together less than two right angles; much more therefore the angles made with  $BC$  by the straight lines which bisect  $ABC$  and  $ACB$  are together less than two right angles. Therefore, by Axiom 12, these straight lines will meet.

280. By I. 47 the square on  $DA$  is equal to the squares on  $DG$  and  $GA$ , and also equal to the squares on  $DE$  and  $EA$ ; therefore the squares on  $DG$  and  $GA$  are equal to the squares on  $DE$  and  $EA$ . But the square on  $DG$  is equal to the square on  $DE$ ; therefore the square on  $GA$  is equal to the square on  $EA$ ; therefore  $GA$  is equal to  $EA$ .

Therefore the angle  $DAG$  is equal to the angle  $DAE$ , by I. 8, so that the angle  $EAG$  is bisected by  $AD$ .

281. Let  $O$  be the centre of the circle inscribed in the triangle  $ABC$ .

The straight line  $OGA$  bisects the angle  $DAE$ , by Exercise 280. Hence the angles  $AOD$  and  $AOE$  are equal, by I. 32. The angle  $ADG$  is equal to the angle  $GED$ , by III. 32; and is therefore half the angle  $DOG$ , by III. 20; that is half the angle  $GOE$ . Also the angle  $GDE$  is half the angle  $GOE$ , by III. 20. Therefore  $DG$  bisects the angle  $ADE$ . Similarly  $EG$  bisects the angle  $AED$ . Therefore  $G$  is the centre of the circle inscribed in the triangle  $ADE$ , by IV. 4.

282. Let it be required to describe a circle which touches the side  $BC$  of a triangle, and also touches the sides  $AB$  and  $AC$  produced.

Produce  $AB$  to any point  $K$ , and  $AC$  to any point  $L$ . Bisect the angles  $KBC$ ,  $LCB$  by the straight lines  $DB$ ,  $DC$  meeting at the point  $D$ . Then  $D$  will be the centre of the required circle. The demonstration will be like that of IV. 4.

In like manner suppose it required to describe a circle which touches the side  $CA$ , and also touches the sides  $BC$  and  $BA$  produced. Produce  $BC$  to any point  $M$ , and  $BA$  to any point  $N$ . Bisect the angles  $ACM$ ,  $CAN$  by the straight lines  $CE$ ,  $AE$  meeting at  $E$ . Then  $E$  will be the centre of the required circle.

We have now to shew that  $D$ ,  $C$ ,  $E$  lie on one straight line.

The angles  $ACB$ ,  $BCL$ ,  $LCM$ ,  $MCA$  are together equal to four right angles, by I. 15, Corollary 1. The angle  $BCD$  is half the angle  $BCL$ , and the angle  $ECA$  is half the angle  $MCA$ , by construction; also the angle  $ACB$  is half the sum of the two angles  $ACB$ ,  $LCM$ , by I. 15. Hence the

angles  $BCD$ ,  $ACB$ ,  $ECA$  are together equal to two right angles. Therefore  $CE$  and  $CD$  are in one straight line, by I. 14.

283. Let the circle which is inscribed in the triangle  $ABC$  touch  $BC$  at  $K$ ; let a circle touch  $BC$  at  $F$ , touch  $AB$  produced at  $D$ , and touch  $AC$  produced at  $E$ . Then  $FK$  will be equal to the difference between  $AB$  and  $AC$ . Suppose  $F$  to fall between  $B$  and  $K$ .

Let the circle inscribed in the triangle  $ABC$  touch  $AB$  at  $G$ , and  $AC$  at  $H$ . By Exercise 176 we have  $AE$  equal to  $AD$ , and  $AH$  equal to  $AG$ , so that  $HE$  is equal to  $GD$ . Now  $HE$  is equal to the sum of  $HC$  and  $CE$ , that is to the sum of  $CK$  and  $CF$ ; and  $GD$  is equal to the sum of  $DB$  and  $BG$ , that is to the sum of  $BF$  and  $BK$ . Hence  $CK$  and  $CF$  are together equal to  $BF$  and  $BK$ ; that is  $KF$  and twice  $CK$  are together equal to  $FK$  and twice  $BF$ : therefore  $BF$  is equal to  $CK$ .

Thus  $FK$ , which is the difference of  $BK$  and  $BF$ , is equal to the difference of  $BK$  and  $CK$ ; that is to the difference of  $BG$  and  $CH$ ; that is to the difference of  $BA$  and  $CA$ . Similarly the result holds if  $F$  falls between  $C$  and  $K$ .

284. Let  $ABC$  be a triangle; let a circle be inscribed in this triangle, touching  $BC$  at  $D$ ,  $CA$  at  $E$ , and  $AB$  at  $F$ . Let a straight line  $GKH$  cut  $AB$  at  $G$ , and  $AC$  at  $K$  and touch the circle at  $H$ ; let a straight line  $LMN$  cut  $BC$  at  $L$ , and  $BA$  at  $N$ , and touch the circle at  $M$ ; let a straight line  $PQR$  cut  $CA$  at  $P$ , and  $CB$  at  $R$ , and touch the circle at  $Q$ . The sides of the three triangles so cut off are together equal to the sides of  $ABC$ .

For, by Exercise 176 we have  $GH$  equal to  $GF$ , so that  $AG$  and  $GH$  together are equal to  $AG$  and  $GF$  together, that is to  $AF$ . Similarly  $BN$  and  $NM$  together are equal to  $BF$ ;  $BL$  and  $LM$  together are equal to  $BD$ ; and so on. Thus the required result will be obtained.

285. The angle  $OBD$  is a right angle; the angle  $DBC$  is half the angle  $ABC$ ; therefore the angle  $OBC$  is the excess of a right angle over half the angle  $ABC$ ; that is half the excess of two right angles over the angle  $ABC$ : therefore  $BO$  bisects the angle between  $BC$  and  $AB$  produced.

Again the angle  $BDO$  is equal to the angles  $DBA$  and  $DAB$ , that is to half the sum of the angles  $ABC$  and  $BAC$ , by IV. 4. The angle  $OBD$  is a right angle, by construction. Therefore the angle  $BOD$  is equal to half the angle  $ACB$ , by I. 32; and is therefore equal to the angle  $BCD$ . Hence a circle will go round  $BDCO$ , by page 276 of the *Euclid*. Therefore the angle  $DCO$  is a right angle by III. 22. Therefore  $CO$  will bisect the angle between  $BC$  and  $AC$  produced.

Hence by the process in the solution of Exercise 282 we see that  $O$  is the centre of the circle which touches the side  $BC$ , and the sides  $AB$ ,  $AC$  produced.

286. Let  $AB$  and  $AC$  produced touch the circle  $GDH$  at  $G$  and  $H$ . Then by Exercise 176  $BG$  is equal to  $BD$ , and  $CH$  is equal to  $CD$ ; also  $AG$  is equal to  $AH$ . Therefore  $AB$  and  $BD$  together are equal to  $AC$  and  $CD$  together; so that  $AB$  and  $BD$  together are equal to half the perimeter of the triangle. Similarly  $BA$  and  $AE$  together are equal to half the perimeter of the triangle. Thus  $AB$  and  $BD$  together are equal to  $BA$  and  $AE$  together. Therefore  $AE$  is equal to  $BD$ . Similarly  $BF$  is equal to  $CE$ , and  $CD$  is equal to  $AF$ .

287. Let  $AB$  and  $AC$  be two straight lines which touch a given circle at



$B$  and  $C$  respectively: it is required to describe a circle which shall touch the two straight lines and the given circle.

Take  $O$  the centre of the given circle. Join  $OA$  cutting the circumference at  $D$ . Draw a straight line through  $D$  at right angles to  $OD$ , and let it meet  $AB$  at  $E$ , and  $AC$  at  $F$ . Bisect the angle  $AED$  by a straight line  $EG$  meeting  $AD$  at  $G$ . Then a circle described from the centre  $G$  with the radius  $GD$  will satisfy the prescribed conditions.

For from the triangles  $AOB$  and  $AOC$  we can shew that  $AG$  bisects the angle  $BAC$ . Hence the circle described from  $G$  as centre with radius  $GD$  will touch the sides of the triangle  $AEF$ , by IV. 4; and it will also touch the circle  $BDC$ .

If  $AO$  produced cut the given circle at  $K$  then another solution can be obtained by using  $K$  instead of  $D$ .

288. Take the diagram of IV. 8. Then  $LA$  is equal to  $LC$ , by Exercise 176; therefore the angle  $LAC$  is equal to the angle  $LCA$ ; therefore each of them is less than a right angle, by I. 17. The angle  $LAC$  is equal to the angle  $ABC$ , by III. 32: therefore the angle  $ABC$  is less than a right angle.

Similarly  $BCA$  and  $CAB$  are each less than a right angle.

289. Let  $ABCD$  be a quadrilateral, in which the sum of the sides  $AB$  and  $CD$  is equal to the sum of the sides  $BC$  and  $AD$ ; and each angle is less than two right angles; then a circle can be inscribed in the quadrilateral.

Bisect the angles  $ABC$  and  $BCD$  by the straight lines  $BO$  and  $CO$  meeting at  $O$ . From  $O$  draw  $OE$  perpendicular to  $BC$ . Then a circle described with  $O$  as centre, and  $OE$  for radius will touch  $AB$ ,  $BC$ , and  $CD$ . This may be shewn by the process of IV. 4. It then remains to shew that it will also touch  $AD$ .

Let  $BA$  and  $CD$  be produced to meet at  $S$ . Suppose if possible that the circle described in the manner prescribed does not touch  $AD$  but cuts it. Suppose a straight line  $LM$  drawn parallel to  $AD$ , cutting  $SA$  at  $L$  and  $SD$  at  $M$  and touching the circle. Then by Exercise 188 the sum of  $LM$  and  $BC$  is equal to the sum of  $LB$  and  $MC$ . But, the sum of  $AD$  and  $BC$  is equal to the sum of  $AB$  and  $DC$ , by supposition. Therefore  $LM$  and  $BC$  are equal to the sum of  $LA$ ,  $AD$ ,  $MD$  and  $BC$ ; that is  $LM$  is equal to the three sides  $AL$ ,  $AD$ ,  $MD$  of the quadrilateral  $ALMD$ . But this is impossible: see I. 20.

Similarly we may shew that the circle described in the manner prescribed cannot fall completely within the quadrilateral  $ABCD$ .

And in like manner we may treat the case in which  $ABCD$  is a parallelogram, so that neither pair of opposite sides will meet when produced.

290. Draw  $SPT$  touching both circles at  $P$ , cutting  $HK$  at  $S$  and  $LM$  at  $T$ . Let  $HL$  cut the straight line which joins the centres of the circles at  $B$ ; and let  $KM$  cut this straight line at  $C$ . They will cut it at right angles.

$LT$  and  $MT$  are each equal to  $PT$ , so that  $T$  is the middle point of  $LM$ . Hence by Exercise 89  $BL$  and  $CM$  are together equal to twice  $PT$ , that is to  $LM$ . In this way we find that  $HL$  and  $KM$  are together equal to  $HK$  and  $LM$ ; and then, by Exercise 289 a circle can be inscribed in  $HKML$ .

To shew that  $HL$  will cut  $BC$  at right angles, see the construction and figure on p. 295 of the *Euclid*.  $ACB$  is a right-angled triangle in which  $AC$  is the difference of the radii of the two circles. When the common tangent is drawn below  $AB$  similarly situated to  $DE$ , the corresponding right-angled

triangle will have its sides equal to those of  $ACB$  by I. 47. Hence the radii from  $A$  to the two points of contact on the circle of which  $A$  is the centre will make equal angles with the line  $AB$ . Then by I. 26 we have the required result.

291. Let  $ABC$  be a triangle. Let  $D$  be the centre of the circle which touches  $BC$ , and also  $AB$  and  $AC$  produced. Let  $E$  be the centre of the circle which touches  $CA$ , and also  $BC$  and  $BA$  produced. Let  $F$  be the centre of the circle which touches  $AB$ , and also  $CA$  and  $CB$  produced. Then  $AD$ ,  $BE$ ,  $CF$  will intersect at the centre of the circle inscribed in the triangle  $ABC$ .

Let  $O$  denote this centre. Then, as in Exercise 280, we can shew that  $OA$  bisects the angle  $BAC$ . In like manner we can shew that  $DA$  bisects the angle  $BAC$ . Thus the straight line  $OA$  coincides in direction with  $DA$ ; so that  $O$  falls on  $DA$ . Similarly  $O$  also falls on  $EB$ , and on  $FC$ .

292. Let  $ABC$  represent the triangle;  $AB$  and  $AC$  being the sides given in position. Describe a circle which touches  $AB$  produced at  $D$ , and  $AC$  produced at  $F$ , and touches  $BC$  at  $E$ . Then, as in Exercise 286, we can shew that  $AD$  and  $AF$  are each equal to half the perimeter of the triangle  $ABC$ ; and so  $D$  and  $F$  are fixed points. Thus the circle touches the fixed straight lines at fixed points, and so it is a fixed circle.

293. Take the diagram of IV. 3. Join  $MK$  and  $KN$ . We shall shew that the angle  $MKN$  is equal to the sum of a right angle together with half the angle  $MLN$ . The angles  $KMA$  and  $KMB$  are equal, so that each of them is half the angle  $LMN$ . Therefore the angle  $MKB$ , half the angle  $LMN$ , and a right angle, are equal to two right angles by I. 32. Similarly the angle  $NKB$ , half the angle  $LMN$ , and a right angle are equal to two right angles. Therefore by addition, the angle  $MKN$ , half the angle  $LMN$ , and half the angle  $LMN$  are equal to two right angles; that is are equal to a right angle together with half the angles at  $L$ ,  $M$ ,  $N$  by I. 32. Therefore the angle  $MKN$  is equal to a right angle together with half the angle at  $L$ . We shall now apply this to the present Exercise. Let  $PQ$  be the given base. Draw a straight line parallel to  $PQ$  and at a distance from it equal to the given radius: then the centre of the inscribed circle will be on this straight line. Again describe on  $PQ$  a segment of a circle containing an angle equal to the sum of a right angle and half the given vertical angle: then from what is shewn above the centre of the inscribed circle will be on this segment. Hence either of the points of intersection of this segment and the straight line drawn parallel to  $PQ$  may be taken as the centre of the inscribed circle: the circle may be drawn, and then tangents to the circle from  $P$  and  $Q$ , produced to meet, will form the other two sides of the required triangle.

#### IV. 5 to 9.

294. From  $F$  draw  $FG$  perpendicular to  $BC$ . Since  $FB$  is equal to  $FC$  the angle  $FBC$  is equal to the angle  $FCB$ ; also the right angle  $FGB$  is equal to the right angle  $FGC$ ; therefore  $BG$  is equal to  $CG$ , by I. 26.

295. The tangent at  $A$  to the circle described round  $ADE$  makes with  $AD$  an angle equal to the angle  $AED$ , by III. 32. The tangent at  $A$  to the

circle described round  $ABC$  makes with  $ADB$  an angle equal to the angle  $ACB$ , by III. 32. But the angle  $AED$  is equal to the angle  $ACB$ , by I. 29. Therefore the tangent at  $A$  to the circle described round  $ABC$  coincides with the tangent at  $A$  to the circle described round  $ADE$ .

296. Let  $ABC$  be a triangle. Let  $O$  be the centre of the circle described about the triangle, and also the centre of the circle inscribed in the triangle: the triangle will be equilateral.

Because  $O$  is the centre of the circle described about the triangle  $O A$ ,  $O B$ ,  $O C$  are all equal. Because  $O$  is the centre of the circle inscribed in the triangle  $O A$ ,  $O B$ , and  $O C$  bisect the angles at  $A$ ,  $B$ , and  $C$  respectively, by IV. 4. Since  $O A$  is equal to  $O B$  the angle  $O A B$  is equal to the angle  $O B A$ ; that is half the angle  $B A C$  is equal to half the angle  $A B C$ : therefore the angle  $B A C$  is equal to the angle  $A B C$ ; therefore  $A C$  is equal to  $B C$ , by I. 6. Similarly  $B A$  is equal to  $C A$ . Thus the triangle  $ABC$  is equilateral.

297. Let  $ABC$  be a triangle; let  $P$  be the centre of the inscribed circle, and  $Q$  the centre of the circumscribed triangle; and suppose that  $P$ ,  $Q$ , and  $A$  are on one straight line: then  $AB$  will be equal to  $AC$ .

From  $Q$  draw  $Q M$  perpendicular to  $AB$ , and  $Q N$  perpendicular to  $AC$ . In the two triangles  $A Q M$  and  $A Q N$  the angle  $Q A M$  is equal to the angle  $Q A N$ ; the right angles  $A M Q$  and  $A N Q$  are equal; and the side  $A Q$  is common: therefore  $A M$  is equal to  $A N$  by I. 26. But  $A M$  is half  $AB$ , and  $A N$  is half  $AC$ , by III. 3: therefore  $AB$  is equal to  $AC$ .

298. Let  $LM$  denote the common chord. The rectangle  $PL$ ,  $PM$  is equal to the square on  $PA$ , by III. 36; the rectangle  $PL$ ,  $PM$  is equal to the rectangle  $PB$ ,  $PC$ , by III. 36 Corollary: therefore the rectangle  $PB$ ,  $PC$  is equal to the square on  $PA$ . Therefore  $PA$  touches the circle which passes through  $A$ ,  $B$ , and  $C$ . Therefore as this circle and the circle  $AML$  have the common tangent  $PA$  they touch each other.

299. Let  $EP$  be the tangent at  $E$  to the circle described round the triangle  $ECD$ , and suppose  $P$  and  $C$  to be on the same side of  $ED$ : then  $EP$  will be parallel to  $AB$ .

The angle  $PEC$  is equal to the angle  $EDC$ , by III. 32. The angle  $EDC$  is equal to the angle  $ABC$ , by III. 22 and I. 13. Therefore the angle  $PEC$  is equal to the angle  $ABC$ . Therefore  $EP$  is parallel to  $AB$ , by I. 27.

300. Let  $A$  and  $B$  be the two given points; join  $AB$ , and produce it to meet the given straight line at  $C$ . On the given straight line take  $D$  such that the square on  $CD$  is equal to the rectangle  $CA$ ,  $CB$ , II. 14. Then the circle which passes through  $A$ ,  $B$ , and  $D$  is the circle required.

For since the rectangle  $CA$ ,  $CB$  is equal to the square on  $CD$  the straight line  $CD$  touches the circle which passes through  $A$ ,  $B$ , and  $D$ , by III. 37.

See the *Euclid*, p. 296.

301. Let  $A$  and  $B$  be the given points; join  $AB$ , and produce it to meet the given straight line at  $C$ . Bisect  $AB$  at  $E$ ; from  $E$  draw a straight line at right angles to  $AB$ ; and in this straight line take any point  $F$  such that  $AF$  is greater than half the given chord. From  $F$  as centre, with radius equal to  $FA$ , describe a circle; this will also pass through  $B$ . Through  $C$  draw a straight line  $CKL$  by Exercise 181 such that the length  $KL$  intercepted by this circle shall be equal to the given chord. On the given straight

line take  $CM$  equal to  $CK$ . Then the circle described round  $A, B$ , and  $M$  will satisfy the prescribed conditions.

For let this circle cut the given straight line at  $N$ . Then the rectangle  $CM, CN$  is equal to the rectangle  $CA, CB$ , by III. 36 Corollary; that is to the rectangle  $CK, CL$ . But  $CM$  is equal to  $CK$ ; therefore  $CN$  is equal to  $CL$ . Therefore  $MN$  is equal to  $KL$ ; that is to the given chord.

302. Let  $AB$  be the given straight line in which the centre is to lie; let  $AC$  and  $BC$  be the other two given straight lines. Bisect the angle  $ACB$  by a straight line meeting  $AB$  at  $D$ . From  $D$  draw a straight line perpendicular to  $AC$  and meeting it at  $E$ ; from  $E$  on the straight line  $EC$  cut off  $EM$  equal to half the given chord. Then the circle described from the centre  $D$ , with the radius  $DM$ , is the required circle.

For let this circle cut  $CA$  again at  $N$ ; then  $MN$  is bisected at  $E$ , by III. 3; therefore  $MN$  is equal to the given chord.

Again, from  $D$  draw a straight line perpendicular to  $BC$  meeting it at  $F$ . Then from the triangles  $CDE$  and  $CDF$  we see that  $DF$  is equal to  $DE$ . Therefore the chord of the circle cut from  $CB$  will be equal to that cut from  $CA$ , by III. 14.

303. Let  $ABC$  and  $KLM$  be two triangles, having the bases  $BC$  and  $LM$  equal, and the angles  $BAC$  and  $LKM$  equal. Let  $F$  be the centre of the circle circumscribing  $ABC$ , and  $G$  the centre of the circle circumscribing  $KLM$ : then  $FB$  will be equal to  $GL$ .

For, since the angle at  $A$  is equal to the angle at  $K$ , the angle  $BFC$  is equal to the angle  $LGM$ , by III. 20; therefore the angles  $FBC$  and  $FCB$  together are equal to the angles  $GLM$  and  $GML$  together; but the angles  $FBC$  and  $FCB$  are equal, and so also are the angles  $GLM$  and  $GML$ , by I. 5. Therefore the angle  $FBC$  is equal to the angle  $GLM$ , and the angle  $FCB$  is equal to the angle  $GML$ . But  $BC$  is equal to  $LM$ , by supposition; therefore  $FB$  is equal to  $GL$ , by I. 26.

304. Let  $A$  be the point from which the tangent is to be drawn; let  $B$  and  $C$  be the two given points through which the circle is to pass. Join  $AB$ ; and in this straight line, produced if necessary, take the point  $D$  such that the rectangle  $AB, AD$  is equal to the square on the given tangent, by I. 45 Corollary. Then  $D$  is also a point through which the circle must pass, by III. 36. Thus we have only to describe a circle passing through the three points  $B, C$ , and  $D$ ; and this can be done by IV. 5.

305. The angle  $APB$  is half the angle  $ACB$ , by III. 20; that is half a right angle; that is equal to  $BAC$ . Therefore  $AB$  touches the circle described round  $ANP$ : see III. 32.

306. Suppose that  $AB$  and  $CD$  are joined towards opposite parts by  $AD$  and  $BC$ . Through  $E$  draw a straight line  $GEH$  touching the circle  $AEB$ . Then the angle  $AEG$  is equal to the angle  $ABE$ , by III. 32. But the angle  $AEG$  is equal to the angle  $DEH$ , by I. 15; and the angle  $ABE$  is equal to the angle  $ECD$ , by I. 29; therefore the angle  $DEH$  is equal to the angle  $ECD$ . Therefore  $GEH$  touches the circle  $ECD$ : see III. 32. Hence the two circles have a common tangent at  $E$ , and therefore touch each other.

So also the proposition may be demonstrated when  $AB$  and  $CD$  are joined towards the same parts by  $AC$  and  $BD$ , and  $AC$  and  $BD$  are produced to meet. See Exercise 295.

307. By III. 14 the centre of the required circle must be equally distant from the three sides of the triangle; and hence it easily follows that it must coincide with the centre of the circle inscribed in the triangle.

308. Join  $BO$ . The angle  $BAF$  is equal to the angle  $CAF$ , by IV. 4. Therefore the arc  $BF$  is equal to the arc  $CF$ , and the straight line  $BF$  to the straight line  $CF$ , by III. 26 and III. 29. Again, the angle  $OBF$  is equal to the angles  $OBC$  and  $FBC$ ; that is to the angles  $OBA$  and  $FAC$ ; that is to the angles  $OBA$  and  $OAB$ ; that is to the angle  $BOF$ , by I. 32. Therefore  $BF$  is equal to  $OF$ , by I. 6.

309. Let  $ABCD$  be the quadrilateral figure; let  $BC$  and  $AD$  be produced to meet at  $P$ , and let  $BA$  and  $CD$  be produced to meet at  $Q$ .

The angles  $DRP$  and  $DCP$  are together equal to two right angles, by III. 22; and so also are the angles  $DRQ$  and  $DAQ$ . Therefore the angles  $DRP$ ,  $DRQ$ ,  $DCP$ , and  $DAQ$  are together equal to four right angles. But the angles  $DCP$  and  $DAQ$  are together equal to two right angles, by I. 13 and III. 22. Therefore the angles  $DRP$  and  $DRQ$  are together equal to two right angles. Therefore  $RP$  and  $RQ$  are in one straight line, by I. 14.

310. Describe a circle round  $ACB$ . Then the straight line which bisects the angle  $ACB$  will bisect the arc cut off by  $AB$ , by III. 26. Again the straight line drawn from the middle point of  $AB$  at right angles to  $AB$  also bisects the arc cut off by  $AB$ : see III. 30. Hence the point  $D$  is on the arc cut off by  $AB$ . Therefore the angles  $ACB$  and  $ADB$  are together equal to two right angles, by III. 22.

311. Draw the tangent to the circle at  $C$ ; and let it meet  $AB$  at  $O$ . Then the angle  $OCE$  is equal to the angle  $CDE$ , by III. 32; that is equal to the angle  $CBA$ , by III. 21. Therefore  $OC$  is equal to  $OB$ , by I. 6; and  $O$  is the centre of the semicircle. Thus the tangent at  $C$  to the circle is at right angles to the tangent at  $C$  to the semicircle.

Similarly the tangent to the circle at  $D$  passes through  $O$ , and the circle cuts the semicircle at right angles at  $D$ .

312. Let  $P, Q, R, S$  be the centres of the circles described round the triangles  $AOB, BOC, COD, DOA$  respectively. Now  $P$  and  $S$  are both on the straight line which passes through the middle point of  $OA$  and is at right angles to  $OA$ , by IV. 5. Thus  $PS$  is at right angles to  $AC$ . Similarly  $QR$  is at right angles to  $AC$ . Therefore  $PS$  is parallel to  $QR$ , by I. 28.

Similarly  $PQ$  is parallel to  $SR$ .

313. The angle  $CED$  is equal to the angle  $ECD$ , by I. 5. But the angle  $ECD$  is equal to the angles  $ECB$  and  $BCD$ ; that is to the angles  $ECB$  and  $BAC$ , by III. 32. And the angle  $CED$  is equal to the angles  $ECA$  and  $BAC$ , by I. 32. Therefore the angles  $ECB$  and  $BAC$  are equal to the angles  $ECA$  and  $BAC$ ; therefore the angle  $ECB$  is equal to the angle  $ECA$ .

314. By the process given we see that  $F$  is the centre of the circle described round the triangle  $ABC$ , so that  $AF$  is equal to  $BF$  and equal to  $CF$ . Now in the triangle  $BFC$  the side  $BC$  is of given length; the angle  $BFC$  is twice the angle  $BAC$ , by III. 20, and is therefore a given angle; the angles  $FBC$  and  $FCB$  are equal, and therefore each of them is given, by I. 32. Hence the triangle  $FBC$  is completely known, and so  $FB$  is of constant length.

315. The angle  $ABD$  is equal to the angle  $ACB$ , by I. 5; and the angle  $ACB$  is equal to the angle  $AEB$ , by III. 21: therefore the angle  $ABD$  is equal to the angle  $BED$ . Therefore  $AB$  touches the circle described round  $BED$ : see III. 32.

316. Let  $E$  be the point of contact of the circles,  $TE$  the common tangent.

Then the angle  $TED$  is equal to the angle  $DBE$ , by III. 32, and the angle  $TEC$  is equal to the angle  $DAE$ ; therefore the angle  $DEC$  is equal to the angle  $AEB$ , by I. 32.

317. Let  $C$  denote the centre of the circle. Then  $HK$  must be as near to  $C$  as possible, by III. 15; so that the angle  $HCK$  must be as great as possible: therefore the angle  $HPK$  must be as great as possible, by III. 20. Thus the problem is reduced to that which is given on page 308 of the *Euclid*.

318. The centres of both circles will lie on the straight line which bisects the angle formed by the two given straight lines. In the given circle place a chord at right angles to this straight line containing the centres, and cutting off a segment containing an angle equal to the given angle. Let  $PQ$  denote this chord. We have finally to describe a circle passing through  $P$  and  $Q$  and touching a given straight line; for this see page 296 of the *Euclid*.

319. Suppose the angles  $ACB$  and  $ABC$  to be acute. Let  $OA$  intersect  $EF$  at  $G$ . The angle  $AOB$  is equal to twice the angle  $ACB$ , by III. 20: therefore the angles  $BAO$  and  $ABO$  are together equal to the excess of two right angles over twice the angle  $ACB$ ; but the angles  $BAO$  and  $ABO$  are equal, by I. 5: therefore the angle  $BAO$  is equal to the excess of a right angle over the angle  $ACB$ .

Again since  $BFC$  and  $BEC$  are right angles a circle will go round  $BFEC$ : see page 276 of the *Euclid*. Therefore the angles  $AFE$  and  $ACB$  are equal, by III. 22 and I. 13.

Thus the angles  $AFG$  and  $FAG$  are together equal to a right angle; and therefore the angle  $AGF$  is a right angle.

The process requires but a slight modification if either of the angles  $ACB$  and  $ABC$  is obtuse.

In like manner it can be shewn that  $OB$  is perpendicular to  $FD$  and  $OC$  to  $DE$ .

320. Let  $ABCD$  be the square,  $P$  any point on the circumference. Since  $ABC$  is a right angle  $AC$  is a diameter of the circle, and  $APC$  is a right angle: see III. 31. Therefore the squares on  $AP$  and  $CP$  are together equal to the square on  $AC$ . Similarly the squares on  $BP$  and  $DP$  are together equal to the square on  $BD$ . Therefore the squares on  $AP$ ,  $BP$ ,  $CP$ ,  $DP$  are together equal to the squares on  $AC$  and  $BD$ , that is to twice the square on  $AC$ .

321. Suppose  $ABCD$  to be a rectangle described about a circle. By Exercise 188 the sum of  $AB$  and  $CD$  is equal to the sum of  $BC$  and  $DA$ ; that is twice  $AB$  is equal to twice  $BC$ : therefore  $AB$  is equal to  $BC$ , and the rectangle is a square.

322. Let  $ABCD$  be a rectangle. Join  $AC$  and  $BD$ , intersecting at  $O$ . By Exercise 78 the diagonals  $AC$  and  $BD$  are bisected at  $O$ ; also  $OB$  is equal

to  $OC$ , by Exercise 59. Thus  $OA, OB, OC, OD$  are all equal; and the circle described from the centre  $O$  with the radius  $OA$  will pass through  $B, C$ , and  $D$ , and thus will be described about the rectangle.

323. Let  $O$  be the centre of the circle; let  $AOB$  and  $COD$  be two diameters. Draw tangents at  $A, B, C, D$  thus forming a quadrilateral figure having for sides  $NAK, KDL, LBM, MCN$ : this quadrilateral figure will be a rhombus.

The straight lines  $MB$  and  $MC$  are equal, by Exercise 176; therefore the angle  $MOB$  is equal to the angle  $MOC$ , by I. 8, so that  $MO$  bisects the angle  $BOC$ ; similarly  $KO$  bisects the angle  $AOD$ : therefore the angle  $MOB$  is equal to the angle  $KOD$ .

In the two triangles  $MOB$  and  $KOD$  the angle  $MOB$  is equal to the angle  $KOD$ ; the angles  $MBO$  and  $KDO$  are equal being right angles; and  $OB$  is equal to  $OD$ ; therefore  $BM$  is equal to  $DK$ . Also  $LB$  is equal to  $LD$ , by Exercise 176. Therefore  $LM$  is equal to  $LK$ .

Similarly it may be shewn that  $LM$  is equal to  $MN$ , and  $MN$  to  $NK$ ; so that the figure  $NKLM$  is a rhombus.

#### IV. 10.

324. The angle  $ACD$  is equal to the two angles  $BDC$  and  $CBD$ , by I. 32. But it is shewn in IV. 10 that the angle  $BDC$  is equal to the angle  $BAD$ , and that the angle  $CBD$  is twice the angle  $BAD$ : therefore the angle  $ACD$  is equal to three times the angle  $BAD$ .

325. The triangle  $BCD$  is shewn in the course of IV. 10 to have each of the angles  $BCD$  and  $CBD$  double of the angle  $CDB$ . Also the triangle  $ACD$  has each of the angles  $CAD$  and  $CDA$  one-third of the angle  $ACD$ : see Exercise 324.

326. Suppose  $F$  the point at which the circles intersect again. Then  $AF$  is equal to  $AD$ . Also the angle  $AFD$  is equal to the angle  $ADB$ , by III. 32; the angle  $ADF$  is equal to the angle  $AFD$ , by I. 5: therefore the angle  $FAD$  is equal to the angle  $BAD$ , by I. 32. Thus the angle  $ADF$  is twice the angle  $DAF$ . Bisect the angle  $ADF$  by the straight line  $DG$  meeting the circumference of the small circle at  $G$ . Then the five angles  $ADG, GDF, FAD, DAC, ADC$  are all equal; and therefore  $CD$  is the side of a regular pentagon inscribed in the small circle.

327. Let  $KL$  be the given base. Make the angles  $MKL$  and  $MLK$  each equal to the angle  $CAD$  of IV. 10. Then the angles  $MKL$  and  $MLK$  are equal to the angles  $CAD$  and  $ADC$ ; therefore the angle  $KML$  is equal to the angle  $ACD$ : therefore the angle  $KML$  is three times the angle  $MKL$  or  $MLK$ . See Exercise 324.

328. This is shewn in the course of the solution of Exercise 326.

329. The angle  $BAG$  is twice the angle  $BAD$ , by Exercise 328; but the angle  $ABD$  is twice the angle  $BAD$ , by IV. 10: therefore the angle  $BAG$  is equal to the angle  $ABG$ , and each of them is twice the angle  $ABD$ .

330. Let  $CA$  produced meet the larger circle at  $G$ , and let  $DC$  produced meet the larger circle at  $H$ : then the triangle  $GCH$  will be of the same kind as the triangle  $ABD$ .

For the angle  $GCH$  is equal to the angle  $BCD$ , by I. 15; and therefore equal to the angle  $ADB$ , by IV. 10. And the angle  $GHD$  is equal to the angle  $GBD$ , by III. 21. Therefore the angle  $HGC$  is equal to the angle  $BAD$ , by I. 32. Therefore each of the angles  $GHC$  and  $HCG$  is double the angle  $HGC$ .

331. The angle  $DAE$  is equal to the angle  $DAC$ ; see Exercise 328: therefore the angle  $DAE$  is equal to the angle  $ADC$ , and therefore  $AE$  is parallel to  $DC$ , by I. 27. The angle  $CDB$  is equal to the angle  $CAD$  by IV. 10; and is therefore equal to the angle  $ECD$ : see Exercise 328: therefore  $BD$  is parallel to  $EC$ , by I. 27. Thus  $CDGE$  is a parallelogram.

332. Let  $E$  denote the point at which the circles cut again. The triangles  $BAD$  and  $EAD$  are equal in all respects by Exercise 328; and the smaller circle is described round the triangle  $EAD$ : therefore an equal circle could be described round the triangle  $BAD$ .

333. Let  $E$  denote the point at which the circles cut again. The angles  $DFE$  and  $DAE$  are together equal to two right angles, by III. 22. The angle  $BAD$  is equal to the angle  $DAE$  by Exercise 328. Therefore the angle  $DFE$  is equal to the excess of two right angles over  $DAB$ ; that is the angle  $DFE$  is equal to the sum of the angles  $CBD$  and  $BDA$ , that is to twice the angle  $CBD$ .

Suppose  $O$  the centre of the circle described round the triangle  $BCD$ . Then the triangles  $OCD$  and  $FDE$  are both isosceles;  $CD$  is equal to  $DE$ ; and the angle  $COD$  by what has been shewn is equal to the angle  $DFE$ . Therefore  $DF$  is equal to  $CO$ , by I. 26.

#### IV. 11 to 16.

334. Let  $ABCDE$  be the regular pentagon; draw  $AC, BD, CE, DA, EB$ . Let  $AC$  and  $BD$  intersect at  $K$ ; let  $BD$  and  $CE$  intersect at  $L$ ; and so on; thus forming the pentagon  $KL MNO$ : this shall be a regular pentagon.

Since  $CB$  is equal to  $CD$  the angle  $CBD$  is equal to the angle  $CDB$ ; so also the angle  $CAB$  is equal to the angle  $BCA$ . But the angle  $ABC$  is equal to the angle  $BCD$ . Therefore by I. 32 the following angles are all equal:  $CBD, CDB, BAC, BCA$ .

Then we can shew that  $BK$  is equal to  $CK$ , that  $CL$  is equal to  $DL$ , and so on. Next by comparing the triangles  $BKC$  and  $CLD$  we can shew that  $CK$  and  $CL$  are equal, and the angle  $BKC$  equal to the angle  $CLD$ ; and so on.

Next by comparing the triangles  $OBK$  and  $KCL$  we can shew that  $OK$  is equal to  $KL$ ; and proceeding thus we find that the pentagon is equilateral. It is also equiangular, since we have the angle  $BKC$  equal to the angle  $CLD$ , and so on; that is the angle  $OKL$  is equal to the angle  $KLM$ , and so on.

335. Suppose a circle described round the pentagon by IV. 14. The angle  $BFC$  is measured by half the sum of the arcs  $AE$  and  $BC$ ; the angle  $FBC$  is measured by half the sum of the arcs  $CD$  and  $DE$ : see page 294 of the *Euclid*. Therefore the angle  $FBC$  is equal to the angle  $BFC$ ; therefore  $BC$  is equal to  $FC$ . And  $AF$  is equal to  $BF$ , since the angle  $ABF$  is equal to the angle  $BAF$ . Thus  $AC$  which is equal to the sum



of  $AF$  and  $FC$  is equal to the sum of  $BF$  and  $CF$ , that is to the sum of  $BF$  and  $AB$ .

336. Let  $ABCDE$  be the regular pentagon. Join  $AC$ ,  $AD$ ; through  $C$  draw a straight line  $CF$  parallel to  $AB$ , meeting  $AD$  at  $F$ .

A circle may be described round the pentagon by IV. 14. Then  $AD$  is parallel to  $BC$ , by Exercise 219; thus  $ABCF$  is a parallelogram; and the triangle  $ABC$  is equal to the triangle  $AFC$ , by I. 34. The triangle  $AED$  is also equal to the triangle  $ABC$ . Thus the regular pentagon exceeds three times the triangle  $ABC$ , namely by the triangle  $FCD$ . Therefore the triangle  $ABC$  is less than a third of the pentagon.

Again  $AC$  is equal to  $AD$ ;  $AB$  and  $BC$  are together greater than  $AC$ , by I. 20; therefore  $AB$  and  $BC$  are together greater than  $AD$ . But  $BC$  is equal to  $AF$ , by I. 34; therefore  $AB$  is greater than  $FD$ ; therefore  $AF$  is greater than  $FD$ . Therefore the triangle  $CFD$  is less than the triangle  $CAF$ . Thus the regular pentagon falls short of four times the triangle  $ABC$ , namely by the excess of the triangle  $ACF$  over the triangle  $CFD$ . Therefore the triangle  $ABC$  is greater than a fourth of the pentagon.

337. Let  $ABC$  be the equilateral triangle,  $O$  the centre of the circle described round it. From  $O$  draw  $OD$  a perpendicular to  $BC$  and produce it to meet the circumference at  $E$ . Then  $BE$  and  $EC$  are sides of the hexagon.

For from the triangles  $BOD$  and  $COD$  we can shew that  $OD$  bisects the angle  $BOC$ . Therefore the angle  $BOD$  is equal to the angle  $BAC$ , by III. 20. Thus the angle  $BOE$  is equal to the angle of an equilateral triangle. But the angles  $OBE$  and  $OEB$  are equal; therefore each of these is the angle of an equilateral triangle. Therefore  $BE$  is equal to  $BO$ .

In this way we can shew that each side of the hexagon is equal to the radius of the given circle; and each angle of the hexagon is equal to twice the angle of an equilateral triangle: thus the hexagon is a regular hexagon.

Also the triangles  $OBC$  and  $OEC$  are equal; and in this way it follows that the regular hexagon is double the given equilateral triangle.

338. Let  $A_1, A_2, A_3, \dots, A_{15}$  be a regular quindecagon inscribed in the given circle. Draw the straight lines  $A_1A_2, A_2A_3, A_3A_4, \dots$ , cutting off arcs which are to one another in the proportion of 2, 5, 8. The angles which stand on these arcs will also be in this proportion; and therefore  $A_1A_2A_3$  will be the triangle required.

339. We can shew by the method used for Exercise 334 that a regular hexagon is obtained; we proceed to shew that the area is one-third of the area of the original hexagon. Let  $ABCDEF$  be the given regular hexagon; let  $O$  be the centre of the circle described about the hexagon.  $O$  will also be the centre of the circle described about the derived hexagon. Let  $FB$  and  $AC$  intersect at  $G$ ; let  $AC$  and  $BD$  intersect at  $H$ .

The angles  $ABG$  and  $BAG$  are equal; so are the angles  $BCH$  and  $HBC$ . Then it may be shewn that all the angles of the triangle  $GBH$  are equal. Thus  $AG, GH, HC$  are all equal. Also  $FC$  is parallel to  $AB$  by Exercise 219; therefore the triangle  $ABC$  is equal to the triangle  $ABO$ .

Again, the angle  $GOH$  will be one-sixth of four right angles, so that  $GOH$  will be an equilateral triangle; therefore it will be equal to the triangle  $GBH$ , that is to one-third of the triangle  $ABC$ , that is to one-third of the triangle  $OAB$ . In this way we can shew that the area of the derived hexagon is one-third of the area of the original hexagon.

340. Let  $A, B, C, D, E \dots$  be consecutive angular points of the equilateral figure inscribed in a circle: this figure will be equiangular. We will shew that the angles  $ABC$  and  $BCD$  are equal.

For, the angle  $ABC$  stands on an arc which consists of the whole circumference except the two smaller arcs cut off by  $AB$  and  $BC$  respectively; and the angle  $BCD$  stands on an arc which consists of the whole circumference except the two smaller arcs cut off by  $BC$  and  $CD$  respectively. Now the smaller arc cut off by  $AB$  is equal to the smaller arc cut off by  $CD$ , by III. 28. Therefore the arc  $ABC$  is equal to the arc  $BCD$ . Hence the arc on which the angle  $ABC$  stands is equal to the arc on which the angle  $BCD$  stands. Therefore the angle  $ABC$  is equal to the angle  $BCD$ , by III. 27.

## VI. 1, 2.

341. In the diagram of IV. 10 take  $E$  on  $AD$  such that  $AE$  may be equal to  $AC$ : then it may be shewn that the triangle  $CDB$  is equal to the triangle  $CAE$ . Therefore the triangle  $CBD$  is to the triangle  $ACD$  as the triangle  $ACE$  is to the triangle  $ACD$ ; that is as  $AE$  is to  $AD$ , by VI. 1; that is as  $AC$  is to  $AB$ . Again, the triangle  $ACD$  is to the triangle  $ABD$  as  $AC$  is to  $AB$ , by VI. 1. Therefore the triangle  $CBD$  is to the triangle  $ACD$  as the triangle  $ACD$  is to the triangle  $ABD$ . Therefore the triangle  $ACD$  is a mean proportional between the triangles  $CBD$  and  $ABD$ .

342. The triangle  $AFE$  is equal to the triangle  $FDE$ , by I. 34; the triangle  $FDE$  is equal to the triangle  $FDC$ , by I. 37: therefore the triangle  $AFE$  is equal to the triangle  $FDC$ . Hence the triangle  $BFD$  is to the triangle  $AFE$  as  $BD$  is to  $DC$ , by VI. 1. Again, the triangle  $AFE$  is to the triangle  $EDC$  as  $AE$  is to  $EC$ , by VI. 1; that is, as  $BD$  is to  $DC$ , by VI. 2. Therefore the triangle  $BFD$  is to the triangle  $AFE$  as the triangle  $AFE$  is to the triangle  $EDC$ ; so that the triangle  $AFE$  is a mean proportional between the triangles  $BFD$  and  $EDC$ .

343. From a point  $O$ , within an equilateral triangle  $ABC$ , let perpendiculars  $OP, OQ, OR$  be drawn on the sides  $BC, CA, AB$  respectively. Also draw  $CD$  from  $C$  perpendicular to  $AB$ . Then in the same manner as in VI. 1. Cor. it may be shewn that triangles on equal bases are to one another as their altitudes; thus the triangle  $OBC$  is to the triangle  $ABC$  as  $OP$  is to  $CD$ ; also the triangle  $OCA$  is to the triangle  $ABC$  as  $OQ$  is to  $CD$ ; and the triangle  $OAB$  is to the triangle  $ABC$  as  $OR$  is to  $CD$ . Hence the sum of the triangles  $OBC, OCA, OAB$  is to the triangle  $ABC$  as the sum of  $OP, OQ, OR$  is to  $CD$ . But the sum of the triangles  $OBC, OCA, OAB$  is equal to the triangle  $ABC$ . Therefore the sum of  $OP, OQ, OR$  is equal to  $CD$ .

344. Let  $ABC$  be a triangle. From  $A$  draw  $AD$  perpendicular to  $BC$ , and from  $D$  on  $DA$  take  $DK$  equal to one-third of  $DA$ . From  $B$  draw  $BE$  perpendicular to  $AC$ , and from  $E$  on  $EB$  take  $EL$  equal to one-third of  $EB$ .

Through  $K$  draw a straight line parallel to  $BC$ ; and through  $L$  draw a straight line parallel to  $AC$ : let these straight lines meet at  $O$ : this shall be the point required.

For the triangles  $ABC$  and  $OBC$  have the same base, but the height of  $OBC$  is one-third of the height of  $ABC$ ; therefore the triangle  $OBC$  is one-

third of the triangle  $ABC$ , as in VI. 1. In like manner the triangle  $OCA$  is one-third of the triangle  $ABC$ . Therefore the triangle  $OAB$  is also one-third of the triangle  $ABC$ .

345.  $CF$  is to  $FB$  as  $AE$  is to  $EB$ , and also  $DG$  is to  $GB$  as  $AE$  is to  $EB$ , by VI. 2. Therefore  $CF$  is to  $FB$  as  $DG$  is to  $GB$ . Therefore  $FG$  is parallel to  $CD$ , by VI. 2.

346. Let  $ABC$  be a triangle. From any point  $K$  in the base  $AB$  draw  $KL$  parallel to  $AC$  meeting  $CB$  at  $L$ , and  $KM$  parallel to  $BC$  meeting  $CA$  at  $M$ .

The diagonals  $CK$  and  $LM$  intersect at  $O$  the middle point of  $CK$ , by Exercise 78. Through  $O$  draw a straight line  $POQ$  parallel to  $AB$ , meeting  $AC$  at  $P$ , and  $BC$  at  $Q$ . Then  $CP$  is to  $PA$  as  $CO$  is to  $OK$ ; but  $CO$  is equal to  $OK$ ; therefore  $CP$  is equal to  $PA$ . Thus  $P$  is the middle point of  $CA$ .

Similarly  $Q$  is the middle point of  $CB$ . Thus  $O$  is on the fixed straight line which joins the middle points of  $AC$  and  $BC$ .

347. Let the straight line from  $D$  parallel to  $BC$  meet  $AC$  at  $E$ . Produce  $AD$  to meet  $BC$  at  $F$ . Then  $AD$  is equal to  $DF$ , by I. 26. And  $AD$  is to  $DF$  as  $AE$  is to  $EC$ , by VI. 2. Therefore  $AE$  is equal to  $EC$ .

348. The triangle  $BED$  is equal to the triangle  $CED$ , by I. 37; therefore the triangle  $DFB$  is equal to the triangle  $EFC$ .

The triangle  $ADF$  is to the triangle  $BDF$  as  $AD$  is to  $DB$ , by VI. 1; that is as  $AE$  is to  $EC$ , by VI. 2; that is as the triangle  $AEF$  is to the triangle  $EFC$ , by VI. 1; that is as the triangle  $AEF$  is to the triangle  $BDF$ . Thus the triangle  $ADF$  is to the triangle  $BDF$  as the triangle  $AEF$  is to the triangle  $BDF$ . Therefore the triangle  $ADF$  is equal to the triangle  $AEF$ .

349. Let  $AF$  produced meet  $BC$  at  $H$ . The triangle  $BFH$  is to the triangle  $BFA$  as  $FH$  is to  $FA$ , by VI. 1; that is as the triangle  $CHF$  is to the triangle  $CFA$ . But from what is shewn in the last Exercise we find that the triangle  $AFB$  is equal to the triangle  $AFC$ . Therefore the triangle  $BFH$  is to the triangle  $BFA$  as the triangle  $CFH$  is to the triangle  $BFA$ . Therefore the triangle  $BFH$  is equal to the triangle  $CFH$ . Therefore  $BH$  is equal to  $CH$ .

350. Let  $ABCD$  be a quadrilateral figure, having the sides  $AB$  and  $DC$  parallel. Let a straight line parallel to these sides meet  $AD$  at  $E$ , and  $BC$  at  $F$ . Then  $DE$  will be to  $EA$  as  $CF$  is to  $FB$ .

Of the two sides  $AB$  and  $DC$  suppose  $AB$  the greater. From  $C$  draw a straight line  $CGH$  parallel to  $DA$  meeting  $EF$  at  $G$ , and  $AB$  at  $H$ . Then  $CE$  and  $GA$  are parallelograms; and therefore  $CG$  is equal to  $DE$ , and  $GH$  is equal to  $EA$ .

Now  $CG$  is to  $GH$  as  $CF$  is to  $FB$ , by VI. 2; therefore  $DE$  is to  $EA$  as  $CF$  is to  $FB$ .

351. Let  $P$  be the given point. Bisect  $PA$  at  $Q$ . From  $Q$  draw a straight line parallel to  $AC$ , meeting  $BC$  at  $R$ . Join  $PR$  and produce it to meet  $AC$ , produced if necessary, at  $S$ .

Then  $PR$  is to  $RS$  as  $PQ$  is to  $QA$ , by VI. 2. But  $PQ$  is equal to  $QA$ ; therefore  $PR$  is equal to  $RS$ .

## VI. 3, A.

352.  $AF$  is to  $FC$  as  $AD$  is to  $DC$ , by VI. 3; and so also  $AE$  is to  $EB$  as  $AD$  is to  $DB$ . But  $DC$  is equal to  $DB$ . Therefore  $AF$  is to  $FC$  as  $AE$  is to  $EB$ . Therefore  $EF$  is parallel to  $BC$ , by VI. 2.

353. The arc  $DBC$  is bisected at  $B$ : see III. 30. Therefore the angle  $DGC$  is bisected by  $GE$ , by III. 27. Therefore  $DG$  is to  $GC$  as  $DE$  is to  $EC$ . Similarly  $DF$  is to  $FC$  as  $DE$  is to  $EC$ . Therefore  $DG$  is to  $GC$  as  $DF$  is to  $FC$ . Therefore also  $DG$  is to  $DF$  as  $GC$  is to  $FC$ , by V. 16.

354. Let  $AB$  be the given straight line. From  $A$  as centre with any radius describe a circle. From  $B$  as centre with a radius equal to double the former describe another circle, cutting the former at  $C$ . Join  $AC$  and  $BC$ , and bisect the angle  $ACB$  by a straight line meeting  $AB$  at  $D$ .

Then  $AC$  is to  $CB$  as  $AD$  is to  $DB$ , by VI. 3. But  $BC$  is twice  $AC$ ; therefore  $DB$  is twice  $AD$ . Bisect  $DB$  at  $E$ . Then  $AD$ ,  $DE$ ,  $EB$  are all equal; so that  $AB$  is trisected.

355. The angle  $CPD$  is bisected by  $PA$ ; therefore  $CA$  is to  $AD$  as  $CP$  is to  $DP$ , by VI. 3. The angle  $APB$  is a right angle, by III. 31; therefore  $PB$  bisects the angle between  $CP$  produced and  $DP$ . Therefore  $CB$  is to  $DB$  as  $CP$  is to  $PD$ , by VI. A. Therefore  $CA$  is to  $AD$  as  $CB$  is to  $DB$ ; therefore  $CA$  is to  $CB$  as  $AD$  is to  $DB$ , by V. 16.

356. On  $AB$  as diameter describe a circle. Bisect the arc  $AB$  at  $C$ . Join  $CD$  and produce it to meet the circumference again at  $E$ . From  $E$  draw a straight line at right angles to  $DE$ , and let it meet  $AB$  produced at  $P$ . Then  $P$  will be the required point.

For, since the arc  $AC$  is equal to the arc  $BC$ , the angle  $AEB$  is bisected by  $ED$ , by III. 27. Therefore  $AE$  is to  $EB$  as  $AD$  is to  $DB$ , by VI. 3. And since the angle  $DEP$  is a right angle  $EP$  bisects the angle between  $AE$  produced and  $BE$ . Therefore  $AE$  is to  $EB$  as  $AP$  is to  $BP$ , by VI. A. Therefore  $AD$  is to  $DB$  as  $AP$  is to  $BP$ .

357.  $AB$  is equal to  $AE$ , by supposition; therefore the angle  $AED$  is equal to the angle  $ABC$ ; also the angle  $EAD$  is equal to the angle  $BAC$ ; therefore  $ED$  is equal to  $BC$ , by I. 26. Now  $AC$  bisects the angle  $BAD$ , and  $AE$  is at right angles to  $AC$ ; therefore  $BC$  is to  $CD$  as  $BA$  is to  $AD$ , by VI. 3; and  $BE$  is to  $ED$  as  $BA$  is to  $AD$ , by VI. A. Therefore  $BE$  is to  $ED$  as  $BC$  is to  $CD$ . But  $ED$  is equal to  $BC$ . Therefore  $BE$  is to  $BC$  as  $BC$  is to  $CD$ ; so that  $BC$  is a mean proportional between  $BE$  and  $CD$ .

358.  $BD$  is to  $DC$  as  $BA$  is to  $AC$ ; therefore the difference of  $BD$  and  $DC$  is to their sum as the difference of  $BA$  and  $AC$  is to their sum: see the *Euclid*, p. 310. Now the difference of  $BD$  and  $DC$  is twice  $DO$ , and their sum is twice  $OB$ . Also twice  $DO$  is to twice  $BO$  as  $DO$  is to  $BO$ , by V. 15. Therefore  $DO$  is to  $BO$  as the difference of  $BA$  and  $AC$  is to their sum.

359. Suppose  $E$  is on  $BC$  produced through  $C$ . By VI. 3 and VI. A we have  $BD$  to  $DC$  as  $BE$  is to  $EC$ . Therefore the difference of  $BD$  and  $DC$  is to  $DC$  as the difference of  $BE$  and  $EC$  is to  $EC$ . That is twice  $OD$  is to  $DC$  as twice  $OC$  is to  $CE$ . Therefore  $OD$  is to  $DC$  as  $OC$  is to  $CE$ . Therefore  $OD$  is to the sum of  $OD$  and  $DC$  as  $OC$  is to the sum of  $OC$  and  $CE$ ; that is  $OD$  is to  $OC$  as  $OC$  is to  $OE$ . But  $OC$  is equal to  $OB$ ; therefore  $OD$  is

to  $OB$  as  $OB$  is to  $OE$ ; so that  $OB$  is a mean proportional between  $OD$  and  $OE$ .

360. Let  $ABC$  be a triangle, let  $D, E, F$  be points in  $BC, CA, AB$ , respectively such that  $DF$  and  $DE$  make equal angles with  $BC, ED$  and  $EF$  make equal angles with  $CA, FE$  and  $FD$  make equal angles with  $AB$ . Then  $AD, BE, CF$  shall be at right angles respectively to  $BC, CA, AB$ .

Let  $AD$  and  $FE$  meet at  $G$ . Since the angles  $GEA$  and  $DEC$  are equal  $AE$  bisects the angle between  $DE$  produced and  $GE$ ; therefore  $DE$  is to  $EG$  as  $DA$  is to  $GA$ . Similarly  $DF$  is to  $FG$  as  $DA$  is to  $GA$ . Therefore  $DE$  is to  $EG$  as  $DF$  is to  $FG$ ; therefore  $DE$  is to  $DF$  as  $EG$  is to  $FG$ . Therefore  $DG$  bisects the angle  $FDE$ , VI. 3. Thus the angle  $GDF$  is equal to the angle  $GDE$ , and the angle  $FDB$  is equal to the angle  $EDC$ ; therefore the angle  $GDB$  is equal to the angle  $GDC$ , so that each of them is a right angle.

Similarly it may be shewn that  $BE$  is at right angles to  $AC$ , and  $CF$  at right angles to  $AB$ .

#### VI. 4 to 6.

361. Let  $ABC$  and  $DEF$  be triangles on equal bases  $AB$  and  $DE$ , and between the same parallels  $ABDE$  and  $CF$ . Let a straight line be drawn parallel to  $ABDE$ , meeting  $AC$  at  $K, BC$  at  $L, DF$  at  $M, EF$  at  $N$ . Then the triangle  $CKL$  shall be equal to the triangle  $FMN$ .

$KL$  is to  $AB$  as  $CL$  is to  $CB$ , and  $MN$  is to  $DE$  as  $FM$  is to  $FD$ , by VI. 4. But  $CL$  is to  $CB$  as  $FM$  is to  $FD$ , by Exercise 350. Therefore  $KL$  is to  $AB$  as  $MN$  is to  $DE$ . Therefore  $KL$  is to  $MN$  as  $AB$  is to  $DE$ . But  $AB$  is equal to  $DE$ , by supposition. Therefore  $KL$  is equal to  $MN$ . Therefore the triangle  $CKL$  is equal to the triangle  $FMN$ , by I. 38.

362.  $CE$  is to  $AB$  as  $FE$  is to  $FB$ , and  $ED$  is to  $AB$  as  $GD$  is to  $GB$ , by VI. 4. But  $CE$  is equal to  $ED$ , by supposition. Therefore  $FE$  is to  $FB$  as  $GD$  is to  $GB$ . Therefore  $FG$  is parallel to  $ED$  by VI. 2.

363. Draw any straight line through  $C$ . Draw  $BN$  and  $AM$  perpendiculars on this straight line. Then  $BN$  is to  $AM$  as  $CB$  is to  $CA$ , by VI. 4; that is  $BN$  has to  $AM$  a constant ratio.

364. Let  $A$  and  $B$  be two fixed points, and suppose a straight line to pass between them and to cut  $AB$  at  $C$ . Draw  $AM$  and  $BN$  perpendiculars on  $MCN$ . Then  $AM$  is to  $AC$  as  $BN$  is to  $BC$ , by VI. 4. Therefore  $AM$  is to  $BN$  as  $AC$  is to  $BC$ . Thus if the ratio of  $AM$  to  $BN$  is given the ratio of  $AC$  to  $BC$  is also given; and therefore  $C$  is a fixed point.

365. Let  $A, B, C$  be three given points. Suppose a straight line to pass through a point  $D$  between  $A$  and  $C$  and also through a point  $E$  between  $B$  and  $C$ . Draw  $AF, CG, BH$  perpendiculars on this straight line. Then  $AF$  is to  $CG$  as  $AD$  is to  $DC$ , by VI. 4. Therefore since the ratio of  $AF$  to  $CG$  is known the point  $D$  is known. Similarly the point  $E$  is known. Thus the required straight line is obtained by joining  $DE$ .

366. Let  $A$  and  $B$  be the points from which the perpendiculars are to be drawn,  $C$  the point through which the straight line is to be drawn.

Join  $AC$  and produce it to  $D$ , making  $AC$  to  $CD$  in the given ratio. Join  $BD$ , and through  $C$  draw a straight line  $ECF$  perpendicular to  $BD$ : then  $ECF$  will be the required straight line.

For draw  $AM$  perpendicular to  $ECF$ , and let  $EF$  and  $BD$  intersect at  $N$ . Then  $CM$  is to  $CN$  as  $AC$  is to  $DC$ , by VI. 4: thus  $CM$  has to  $CN$  the assigned ratio.

367. The angle  $BFD$  is equal to the angle  $EFC$ , by I. 15; the angle  $DBF$  is equal to the angle  $FEC$  by I. 29. Therefore the triangle  $BFD$  is equiangular to the triangle  $EFC$ . Therefore  $BD$  is to  $CE$  as  $BF$  is to  $FE$ . But  $BD$  is equal to  $BA$ , and  $CA$  is equal to  $CE$ ; therefore  $BA$  is to  $AC$  as  $BF$  is to  $FE$ . Therefore  $AF$  is parallel to  $CE$ , by VI. 2.

368. From  $P$  draw  $PY$  perpendicular to  $AB$ , and from  $Q$  draw  $QZ$  perpendicular to  $CD$ . Then the triangle  $PMY$  is equiangular to the triangle  $QNZ$ ; therefore  $PM$  is to  $QN$  as  $PY$  is to  $QZ$ . Thus the ratio of  $PM$  to  $QN$  is constant.

Again, let  $NM$  and  $QP$  produced if necessary meet at  $R$ . Then the triangle  $PMR$  is equiangular to the triangle  $QNR$ ; therefore  $RP$  is to  $RQ$  as  $PM$  is to  $QN$ ; so that the ratio of  $RP$  to  $RQ$  is constant: therefore  $R$  is a fixed point.

369. Let  $ABCD$  be a quadrilateral figure, in which  $AB$  is parallel to  $CD$  and equal to twice  $CD$ . Join  $AC$  and  $BD$  intersecting at  $O$ . Then  $OC$  will be one-third of  $AC$ .

The angle  $DOC$  is equal to the angle  $BOA$ , by I. 15; the angle  $OCD$  is equal to the angle  $OAB$ , by I. 29; therefore the triangle  $DOC$  is equiangular to the triangle  $BOA$ . Therefore  $AO$  is to  $CO$  as  $AB$  is to  $DC$ , by VI. 4. But  $AB$  is twice  $DC$ ; therefore  $AO$  is twice  $CO$ . Bisect  $AO$  at  $P$ ; then  $AP$ ,  $PO$ ,  $OC$  are all equal and  $OC$  is one-third of  $AC$ .

370. The angles  $CAT$  and  $CBT$  are right angles; therefore a circle will go round  $CATB$ : see page 276 of the *Euclid*. Therefore the angle  $CAB$  is equal to the angle  $CTB$ , by III. 21; therefore the angle  $ABN$  is equal to the angle  $CTB$ . Also the right angles  $ANB$  and  $CBT$  are equal. Therefore the angle  $NAB$  is equal to the angle  $BCT$ , by I. 32. Thus the triangle  $BAN$  is equiangular to the triangle  $BCT$ ; and therefore  $BT$  is to  $BC$  as  $BN$  is to  $NA$ , by VI. 4.

371. Through  $E$  draw  $OE$  parallel to  $AB$ , meeting  $BC$  at  $O$ . Then  $AB$  is to  $AC$  as  $OE$  is to  $EC$ , by VI. 4; that is as  $OE$  is to  $BD$ , by supposition; that is as  $EF$  is to  $DF$ , by VI. 4.

372. Let  $P$  be the centre of the circle which passes through  $A$ ,  $C$ , and any point  $D$  in  $BC$ ; let  $Q$  be the centre of the circle which passes through  $A$ ,  $B$ , and  $D$ .

The angle  $APC$  is equal to twice the excess of two right angles over  $ADC$  by III. 22, that is the angle  $APC$  is twice  $ADB$ ; also the angle  $AQB$  is twice  $ADB$ , by III. 20; therefore the angle  $AQB$  is equal to the angle  $APC$ . Thus the isosceles triangle  $AQB$  is equiangular to the isosceles triangle  $APC$ ; and therefore  $PA$  is to  $QA$  as  $AC$  is to  $AB$ .

373. Let  $ABC$  be a triangle. Suppose that the perpendicular from the required point on  $BC$  is to be to the perpendicular on  $CA$  in the ratio of  $X$  to  $Y$ .

Draw a straight line parallel to  $BC$  at the distance  $X$  from it; and draw a straight line parallel to  $CA$  at a distance  $Y$  from it; let these straight lines meet at  $D$ . Then the perpendicular from  $D$  on  $BC$  is equal to  $X$ , and the perpendicular from  $D$  on  $CA$  is equal to  $Y$ .

Join  $CD$ ; take any point  $P$  on  $CD$ , and draw  $PM$  perpendicular to  $BC$  and  $PN$  perpendicular to  $CA$ . Then  $PM$  is to  $X$  as  $CP$  is to  $CD$ , by VI. 4; and also  $PN$  is to  $Y$  as  $CP$  is to  $CD$ : therefore  $PM$  is to  $PN$  as  $X$  is to  $Y$ .

Again, suppose that the perpendicular from the required point on  $CA$  is to be to the perpendicular on  $AB$  as  $Y$  is to  $Z$ . Then as before we find a point  $E$  such that the perpendicular from it on  $AC$  is  $Y$ , and the perpendicular from it on  $AB$  is  $Z$ . Join  $AE$ ; then, as before, we can shew that the perpendiculars from any point in  $AE$  on  $CA$  and  $AB$  are in the ratio of  $Y$  to  $Z$ .

Let  $CD$  and  $AE$  be produced to meet at  $O$ . Then from what has been shewn it follows that the perpendiculars from  $O$  on  $BC$ ,  $CA$ ,  $AB$  are proportional to  $X$ ,  $Y$ ,  $Z$  respectively.

374. Let  $AEB$  be one triangle, and  $AFC$  the other. Suppose that  $AE$  and  $AF$  are homologous. Draw  $EM$  perpendicular to  $AB$ , and  $FN$  perpendicular to  $AC$ ; and produce  $EM$  and  $FN$  to meet at  $P$ . Let  $D$  be the angle of the rectangle opposite to  $A$ .

Then the triangle  $AEM$  is equiangular to the triangle  $AFN$ ; and therefore  $AM$  is to  $AN$  as  $AE$  is to  $AF$ , that is as  $AB$  is to  $AC$ . Therefore  $AM$  is to  $MP$  as  $AB$  is to  $BD$ ; therefore the triangle  $AMP$  is equiangular to the triangle  $ABD$ , by VI. 6; therefore  $P$  is on the straight line  $AD$ .

If  $EB$  and  $AF$  are homologous it will be found that the perpendiculars meet on  $BC$ .

375. Let  $GE$  produced through  $E$ , and  $CA$  produced through  $A$ , meet at the point  $P$ . Then  $CP$  is to  $KP$  as  $CG$  is to  $KE$ , by VI. 4. Let  $FH$  produced through  $H$ , and  $CA$  produced through  $A$ , meet at the point  $Q$ . Then  $CQ$  is to  $KQ$  as  $CF$  is to  $KH$  by VI. 4. But the triangle  $CGK$  is equiangular to the triangle  $KEA$ ; therefore  $CG$  is to  $GK$  as  $KE$  is to  $EA$ , that is  $CG$  is to  $CF$  as  $KE$  is to  $KH$ ; therefore  $CG$  is to  $KE$  as  $CF$  is to  $KH$ .

Therefore  $CP$  is to  $KP$  as  $CQ$  is to  $KQ$ : therefore the points  $P$  and  $Q$  coincide; so that  $GE$ ,  $FH$ , and  $CA$ , produced, meet at a point.

376. Let  $PQ$  and  $AC$ , produced if necessary, meet at  $L$ . Then  $LP$  is to  $LQ$  as  $AP$  is to  $CQ$ , by VI. 4. Let  $PQ$  and  $BD$ , produced if necessary, meet at  $M$ . Then  $MP$  is to  $MQ$  as  $PB$  is to  $QD$ . Now  $AP$  is  $PB$  as  $CQ$  is to  $QD$ , by supposition. Therefore  $LP$  is to  $LQ$  as  $MP$  is to  $MQ$ . Therefore  $L$  coincides with  $M$ .

If instead of having given that  $AP$  is to  $PB$  as  $CQ$  is to  $DQ$  we have  $AP$  is to  $PB$  as  $DQ$  is to  $QC$ , we can shew that  $PQ$ ,  $AD$ ,  $BC$  meet at a point.

377. Let the straight line parallel to  $AB$  cut  $AC$  at  $M$ , and  $BC$  at  $N$ . From  $M$  and  $N$  draw straight lines parallel to  $BD$  meeting  $AB$  at  $P$  and  $Q$  respectively. Then  $PA$  will be equal to  $QB$ .

From  $C$  draw a straight line parallel to  $AB$  meeting  $BD$  at  $E$ . Then since the straight line  $AD$  is bisected at  $C$  the straight line  $BD$  is bisected at  $E$ , by VI. 2.

The triangle  $MAP$  is equiangular to the triangle  $DCE$ ; so that  $MP$  is to  $PA$  as  $DE$  is to  $EC$ , by VI. 4. Again, the triangle  $NBQ$  is equiangular

to the triangle  $BCE$ , so that  $NQ$  is to  $QB$ , as  $BE$  is to  $EC$ , by VI. 4; that is as  $DE$  is to  $EC$ .

Therefore  $MP$  is to  $PA$  as  $NQ$  is to  $QB$ ; therefore  $MP$  is to  $NQ$  as  $PA$  is to  $QB$ . But  $MP$  is equal to  $NQ$ , by I. 34; therefore  $PA$  is equal to  $QB$ .

378. Let  $B$  and  $C$  be the centres of the two given circles. Let a circle having its centre at  $A$  touch externally the former circle at  $R$ , and the latter at  $S$ . The straight lines  $RS$  and  $BC$  when produced will meet at a fixed point.

Let the straight line  $RS$  intersect the circle with centre  $C$  again at  $P$ , and let it meet the line  $BC$  at  $T$ . The angle  $CPS$  is equal to the angle  $OSP$  by I. 5; and is therefore equal to the angle  $ASR$ , by I. 15; and is therefore equal to the angle  $ARS$ , by I. 5.

Therefore  $AB$  is parallel to  $CP$ , by I. 27. Therefore  $TC$  is  $TB$  as  $CP$  is to  $BR$ , by VI. 4. Thus the ratio of  $TC$  to  $TB$  is a fixed quantity; and therefore  $T$  is a fixed point.

379. From  $E$  and  $F$  draw straight lines parallel to  $AD$  meeting  $BC$  at  $M$  and  $N$  respectively.

Then  $EM$  is to  $MC$  as  $AD$  is to  $DC$ , by VI. 4; that is as  $AD$  is to  $DB$  by supposition; that is as  $FN$  is to  $NB$ , by VI. 4. Therefore  $EM$  is to  $FN$  as  $MC$  is to  $NB$ .

Again,  $EM$  is to  $MB$  as  $PD$  is to  $DB$ , by VI. 4; that is as  $PD$  is to  $DC$ , by supposition; that is as  $FN$  is to  $NC$ , by VI. 4.

Therefore  $EM$  is to  $FN$  as  $MB$  is to  $NC$ .

Therefore  $MC$  is to  $NB$  as  $MB$  is to  $NC$ ; therefore  $MC$  is to  $MB$  as  $NB$  is to  $NC$ . Therefore  $MC$  is to the sum of  $MC$  and  $MB$  as  $NB$  is to the sum of  $NB$  and  $NC$ . That is  $MC$  is to  $CB$  as  $NB$  is to  $CB$ ; therefore  $MC$  is equal to  $NB$ .

But  $EM$  is to  $FN$  as  $MC$  is to  $NB$ . Therefore  $EM$  is equal to  $FN$ . Therefore  $FE$  is parallel to  $NM$ , by I. 33.

380. Let  $S$  be the centre of the circle of which  $AE$  is a diameter, and  $T$  the centre of the circle of which  $EB$  is a diameter. Then  $SL$  is to  $SP$  as  $TL$  is to  $TQ$ , by VI. 4; therefore the difference of  $SL$  and  $SP$  is to  $SP$  as the difference of  $TL$  and  $TQ$  is to  $TQ$ , by V. 17; that is  $EL$  is to  $SP$  as  $BL$  is to  $TQ$ . Now  $SP$ , being equal to  $SE$ , is three times  $ET$ , that is three times  $TQ$ ; therefore  $EL$  is three times  $BL$ ; therefore  $EB$  is twice  $BL$ ; therefore  $BL$  is equal to  $BT$ .

381. The triangle  $AOE$  is equiangular to the triangle  $AED$ ; for the angle  $EAO$  is common to the two triangles, and the angle  $AEO$  is equal to the angle  $ADE$  by IV. 14 and III. 27, and therefore the angle  $AOE$  is equal to the angle  $AED$ , by I. 32. Therefore  $AO$  is to  $AE$  as  $AE$  is to  $AD$ ; so that  $AE$  is a mean proportional between  $AO$  and  $AD$ .

382.  $QR$  is to  $BR$  as  $PQ$  is to  $AB$ , by VI. 4; that is as  $PQ$  is to  $DC$ , by I. 34; that is as  $SQ$  is to  $CS$ , by VI. 4. Thus  $QR$  is to  $BR$  as  $QS$  is to  $CS$ ; therefore  $RS$  is parallel to  $BC$ , by VI. 2.

383.  $BD$  is to  $EF$  as  $CD$  is to  $CF$ , by VI. 4; and also  $EF$  is to  $AC$  as  $FD$  is to  $CD$ : therefore  $BD$  is to  $AC$  as  $FD$  is to  $CF$ , by V. 23. Therefore  $BD$  is to  $FD$  as  $AC$  is to  $CF$ . Therefore the triangle  $BDF$  is equiangular to the triangle  $ACF$ , by VI. 6: therefore the angle  $BFD$  is equal to the angle  $AFC$ .



384. From  $A$  draw  $AE$  perpendicular to  $BD$ , from  $B$  draw  $BF$  perpendicular to  $AC$ , from  $C$  draw  $CG$  perpendicular to  $BD$ , and from  $D$  draw  $DH$  perpendicular to  $AC$ . Let  $AC$  and  $BD$  intersect at  $O$ .

The triangle  $OAE$  is equiangular to the triangle  $OFB$ , and therefore  $OA$  is to  $OE$  as  $OB$  is to  $OF$ ; therefore the triangle  $OFE$  is equiangular to the triangle  $OAB$  by VI. 6; therefore the angle  $OEF$  is equal to the angle  $OAB$  and the angle  $OFE$  to the angle  $OBA$ . Similarly the angle  $OEH$  is equal to the angle  $OCB$ , that is to the angle  $OAD$ . Therefore the angle  $FEH$  is equal to the angle  $BAD$ . And the angle  $EFH$  has been shewn equal to the angle  $ABD$ . Therefore the angle  $FHE$  is equal to the angle  $BDA$ . Therefore the triangle  $FEH$  is equiangular to the triangle  $BAD$ ; and therefore similar to it, by VI. 4.

Similarly the triangle  $FGH$  is similar to the triangle  $BCD$ . Hence it will follow that  $EFGH$  is similar to  $ABCD$ .

385. Let  $A$  be the centre of one circle, and  $C$  that of the other. Let them intersect at the given point  $B$ . Then, by supposition,  $BA$  and  $BC$  are fixed directions. Let a straight line touch the former circle at  $E$ , and the latter at  $F$ ; and let  $EF$  produced meet  $AC$  produced at  $D$ . Then  $D$  is the point at which two tangents to both circles will intersect.

Then  $DA$  is to  $DC$  as  $AE$  is to  $CF$ , by VI. 4; that is as  $AB$  is to  $BC$ . Therefore  $DB$  bisects the angle between  $AB$  produced and  $BC$ , by VI. A: thus  $D$  is on a fixed straight line.

Now as  $BA$  and  $BC$  are fixed directions other cases may occur in which instead of  $A$  we have some point on  $AB$  produced through  $B$ , or in which we have instead of  $C$  some point on  $CB$  produced through  $B$ . Thus we obtain a second fixed straight line, namely that which bisects the angle  $ABC$ .

#### VI. 7 to 18.

386. Let the circles touch each other at the point  $B$ ; let one circle touch the straight line at  $C$ , and let the other circle touch the straight line at  $D$ . Draw  $CA$  a diameter of the former circle, and  $DE$  a diameter of the latter. Draw the straight line  $BF$  touching the circles at  $B$ , and meeting  $CD$  at  $F$ .

Thus  $FB$ ,  $FC$ ,  $FD$  are all equal, by Exercise 176. Therefore a circle described with  $F$  as centre will go through  $B$ ,  $C$ , and  $D$ ; therefore  $CBD$  will be a right angle, by III. 31. Also  $EBD$  is a right angle, by III. 31. Thus  $CBE$  is a straight line, by I. 14. Similarly  $DBA$  is a straight line.

Now the angle  $BCD$  is equal to the angle  $CAB$ , and the angle  $BDC$  is equal to the angle  $DEB$ , by III. 32. Therefore the triangle  $ACD$  is equiangular to the triangle  $CDE$ ; therefore  $ED$  is to  $DC$  as  $DO$  is to  $CA$ .

387. Let  $EDK$  be the given arc, and  $EGK$  the remaining part of the circumference. Bisect the arc  $EGK$  at  $F$ . Divide  $EK$  at  $H$  so that  $EH$  may be to  $HK$  in the given ratio. Join  $FH$  and produce it to meet  $EDK$  at  $L$ : then  $L$  will be the point required.

For since the arc  $EF$  is equal to the arc  $FK$  the angle  $ELF$  is equal to the angle  $FLK$ , by III. 27. Therefore  $EL$  is to  $LK$  as  $EH$  is to  $HK$  by VI. 3: so that  $EL$  is to  $LK$  in the given ratio.

388. Let  $ABC$  be the triangle; draw  $CE$  parallel to  $AB$ , and make  $CE$  to  $CB$  as  $CB$  is to  $BA$ . Join  $BE$  cutting  $AC$  at  $D$ . Draw  $DF$  parallel to  $AB$ , meeting  $BC$  at  $F$ .

Then  $DF$  is to  $FB$  as  $CE$  is to  $CB$ , by VI. 4; that is as  $CB$  is to  $BA$ , by construction; that is as  $CF$  is to  $FD$ , by VI. 4. Thus  $DF$  is to  $FB$  as  $CF$  is to  $FD$ ; therefore  $FB$  is to  $FD$  as  $FD$  is to  $CF$ , by V. B. Thus  $FD$  is a mean proportional between  $FB$  and  $FC$ .

389. The angles  $BDA$  and  $ADC$  are equal being right angles; also  $BD$  is to  $DA$  as  $DA$  is to  $DC$ , by supposition. Therefore the triangle  $BDA$  is equiangular to the triangle  $ADC$ , by VI. 6. Thus the angle  $BAD$  is equal to the angle  $ACD$ , and the angle  $ABD$  is equal to the angle  $CAD$ . Therefore the angle  $BAC$  is equal to the two angles  $ABC$  and  $ACB$ ; therefore the angle  $BAC$  is a right angle.

390. Since  $BD$  is to  $BA$  as  $BA$  is to  $BC$  the triangle  $BDA$  is equiangular to the triangle  $BAC$ , by VI. 6; therefore the angle  $BAC$  is equal to the angle  $BDA$ ; that is the angle  $BAC$  is a right angle.

391.  $CA$  is to  $CP$  as  $CP$  is to  $CB$ , by supposition; therefore the triangle  $ACP$  is equiangular to the triangle  $PCB$ , by VI. 6; therefore the angle  $CPA$  is equal to the angle  $CBP$ .

392. Let the centre of the circle in any position be at a point  $C$ , and let the circle touch the straight line  $OA$  at the point  $B$ .

$PQ$  is a third proportional to  $OP$  and  $PC$ ; so that  $OP$  is to  $PC$  as  $PC$  is to  $PQ$ ; therefore the angle  $OCQ$  is a right angle by Exercise 389. Produce  $QC$  to meet  $OA$  at  $N$ . Then in the triangles  $OCQ$ ,  $OCN$  the angle  $COQ$  is equal to the angle  $CON$  because the triangles  $COP$ ,  $COB$  are equal by Exercise 176; the right angle  $OCQ$  is equal to the right angle  $OCN$ ; and  $OC$  is common: therefore  $QC$  is equal to  $CN$ , by I. 26.

From  $Q$  draw  $QM$  perpendicular to  $OA$ . Then the triangle  $QNM$  is equiangular to the triangle  $CNB$ ; therefore  $QM$  is to  $CB$  as  $QN$  is to  $CN$ , by VI. 4. But  $QN$  has been shewn to be twice  $CN$ ; therefore  $QM$  is twice  $CB$ . Thus  $Q$  is always on a straight line which is parallel to  $OA$  and at a distance from it equal to twice the given radius.

393. Let  $AS$  and  $BT$  be the parallel straight lines,  $AB$  being a diameter of the circle; let  $C$  be the centre of the circle. Then  $SCT$  is a right angle, by Exercise 182. Therefore  $CP$  is a mean proportional between  $SP$  and  $PT$ , by VI. 8, Corollary. Therefore the rectangle  $SP$ ,  $PT$  is equal to the square on  $CP$ , by VI. 17; thus the rectangle  $SP$ ,  $PT$  is constant.

394. Suppose  $D$  the point in the side  $AB$  of the triangle  $ABC$ , and let  $DE$  be parallel to  $BC$ . Then the triangle  $ADE$  must be equal to the triangle  $DBC$ ; therefore  $AD$  is to  $DB$  as  $BC$  is to  $DE$ , by VI. 15. But  $BC$  is to  $DE$  as  $AB$  is to  $AD$ , by VI. 4. Therefore  $AD$  is to  $DB$  as  $AB$  is to  $AD$ ; therefore the rectangle  $AB$ ,  $DB$  must be equal to the square on  $AD$ . Thus  $AB$  must be divided at  $D$  in the manner of II. 11.

395. The triangles  $ECA$  and  $BCD$  are equiangular; therefore  $EC$  is to  $CA$  as  $CB$  is to  $CD$ , by VI. 4; therefore the triangle  $ECD$  is equal to the triangle  $ACB$ , by VI. 15.

396. The triangle  $ABE$  is equiangular to the triangle  $CBF$ ; therefore  $AB$  is to  $BE$  as  $CB$  is to  $BF$ , by VI. 4; therefore  $AB$  is to  $CB$  as  $BE$  is to  $BF$ ; therefore the triangle  $ABF$  is equal to the triangle  $CBE$ , by VI. 15.

397. Let  $ABCD$  be a quadrilateral figure inscribed in a circle; let  $AC$  and  $BD$  intersect at  $O$ .

The angle  $AOD$  is equal to the angle  $BOC$ , by I. 15; the angle  $DAO$  is equal to the angle  $CBO$ , by III. 21; and the angle  $ADO$  is equal to the angle  $BCO$ , by III. 21; therefore the triangle  $AOD$  is equiangular to the triangle  $BOC$ . Therefore  $DO$  is to  $AO$  as  $CO$  is to  $BO$ , by VI. 4. Therefore the rectangle  $DO, OB$  is equal to the rectangle  $CO, OA$ , by VI. 16.

Similarly it may be shown that the triangle  $COD$  is equiangular to the triangle  $BOA$ .

398. Let  $EF$  and  $CD$  meet at  $M$ . Then  $GO$  is to  $EM$  as  $CO$  is to  $CM$ , and  $LO$  is to  $FM$  as  $CO$  is to  $CM$ , by VI. 4. Therefore  $GO$  is to  $EM$  as  $LO$  is to  $FM$ . Therefore  $GO$  is to  $LO$  as  $EM$  is to  $FM$ . Similarly  $KO$  is to  $HO$  as  $EM$  is to  $FM$ . Therefore  $GO$  is to  $LO$  as  $KO$  is to  $HO$ . Therefore the rectangle  $GO, HO$  is equal to the rectangle  $LO, KO$ , by VI. 16.

399. The angles  $DFC, ACD, BDC$  and half of  $ACB$  make up two right angles, by I. 32. The angles  $DGC, ACD, BDC$  and half of  $ADB$  also make up two right angles. Therefore the angles  $DFC$  and half of  $ACB$  are together equal to the angles  $DGC$  and half of  $ADB$ . Therefore the angle  $DFC$  is equal to the angle  $DGC$ , by III. 21. Therefore a circle will go round  $DGFC$ , by page 276 of the *Euclid*. Therefore the rectangle  $EG, ED$  is equal to the rectangle  $EF, EC$ , by III. 36, Corollary. Therefore  $EF$  is to  $EG$  as  $ED$  is to  $EC$ , by VI. 16.

400. Let  $ABC$  be the triangle. From  $A$  draw a straight line  $AD$ , meeting  $BC$  at  $D$ ; and also draw from  $A$  a straight line  $AE$ , meeting the circumference of the circumscribing circle at  $E$ , such that the angle  $ACE$  is equal to the angle  $ADB$ .

Then the angle  $ABD$  is equal to the angle  $AEC$ , by III. 21; the angle  $ADB$  is equal to the angle  $ACE$ , by construction; therefore the angle  $BAD$  is equal to the angle  $EAC$ , by I. 32. Thus the triangle  $BAD$  is equiangular to the triangle  $EAC$ ; and therefore  $AB$  is to  $AD$  as  $AE$  is to  $AC$ , by VI. 4; and therefore the rectangle  $AB, AC$  is equal to the rectangle  $AD, AE$ .

401.  $AC$  is to  $CE$  as  $CD$  is to  $CB$ ; and the angle  $ACE$  is equal to the angle  $DCB$ ; therefore the triangle  $ACE$  is equiangular to the triangle  $DCB$ , by VI. 6, so that the angle  $CEA$  is equal to the angle  $CBD$ . Therefore a circle will go round  $CBEA$ ; and the point  $E$  will bisect the arc  $AEB$ , because the angle  $ACE$  is equal to the angle  $BCE$ . If  $AB$  and the angle  $ACB$  are given, this circle will be a fixed circle, and  $E$  will be a fixed point on the circumference. See p. 276 of the *Euclid*.

402. Let  $DFGE$  be the square;  $F$  being on the side  $AC$ , and  $G$  on the side  $BC$ . Then the triangle  $ADF$  is equiangular to the triangle  $GEB$ ; therefore  $EB$  is to  $EG$  as  $DF$  is to  $DA$ . Therefore the rectangle  $AD, BE$  is equal to the rectangle  $EG, DF$ , by VI. 16; that is to the rectangle  $EG, ED$ ; that is to the square  $DFGE$ .

403. The triangle  $AFE$  is equiangular to the triangle  $CFB$ ; therefore  $EF$  is to  $FB$  as  $FA$  is to  $FC$ , by VI. 4. In like manner from the triangles  $GFC$  and  $BFA$  we have  $FB$  to  $FG$  as  $FA$  to  $FC$ . Therefore  $EF$  is to  $FB$  as  $FB$  is to  $FG$ ; therefore the rectangle  $EF, FG$  is equal to the square on  $FB$ , by VI. 17.

404. In the triangle  $ABC$  suppose that  $AB$  is equal to  $AC$ . From  $A$  draw a straight line meeting  $BC$  at  $D$ , and produce  $AD$  to meet at  $E$  the circumference of the circle described round  $ABC$ .

The angle  $AEB$  is equal to the angle  $ACB$ , by III. 21; the angle  $ACB$  is equal to the angle  $ABC$ , by I. 5; therefore the angle  $AEB$  is equal to the angle  $ABD$ . Thus the triangle  $AEB$  is equiangular to the triangle  $ABD$ , by I. 32. Therefore  $DA$  is to  $AB$  as  $AB$  is to  $AE$ , by VI. 4; therefore the rectangle  $DA, AE$  is equal to the square on  $AB$ , by VI. 17.

405. Let  $T$  be one of the points of contact of the given tangents, and  $P$  one of the points of intersection of the two circles. Then the square on  $ET$  is equal to the rectangle  $EA, EH$ , by VI. 8, Corollary; and  $ET$  is equal to  $EP$ ; therefore the square on  $EP$  is equal to the rectangle  $EA, EH$ ; therefore  $EP$  touches the circle  $HPA$  by III. 37.

## VI. 19 to D.

406. In the diagram of IV. 10 suppose a straight line drawn from  $C$  parallel to  $BD$ , meeting  $AD$  at  $F$ . Then  $FB$  bisects the angle  $ABD$ : see Exercise 63. The triangle  $ACF$  will be to the figure  $BCFD$  as  $BD$  is to  $BA$ .

The triangle  $ACF$  is equiangular to the triangle  $ABD$ ; and therefore the triangle  $ACF$  is to the triangle  $ABD$  as the square on  $AC$  is to the square on  $AB$ , by VI. 19; that is as the rectangle  $AB, BC$  is to the square on  $AB$ , by IV. 10; that is  $BC$  is to  $AB$ . Therefore the triangle  $ACF$  is to the figure  $BCFD$  as  $BC$  is to  $AC$ , see V. E. But  $AC$  is equal to  $BD$ ; therefore the triangle  $ACF$  is to the figure  $BCFD$  as  $BC$  is to  $BD$ ; that is as  $BD$  is to  $BA$ , for the triangle  $BCD$  is equiangular to the triangle  $BAD$ .

407. Let  $EB$  be a side of the regular polygon,  $K$  the centre of the circle. Let  $CB$  be half the side of the circumscribed figure of half the number of sides;  $C, E, K$  being in one straight line. From  $E$  draw  $EL$  perpendicular to  $BK$ ; then  $EL$  is half the side of the inscribed figure of half the number of sides.

Let  $X, Y, Z$  denote the areas of the three figures respectively in descending order of magnitude. Then  $X$  is to  $Z$  as the triangle  $CBK$  is to the triangle  $ELK$ ; that is in the duplicate ratio of  $CB$  to  $EL$ , by VI. 19. Also  $X$  is to  $Y$  as the triangle  $CBK$  is to the triangle  $EBK$ ; that is as  $CK$  is to  $EK$ , by VI. 1; that is as  $CB$  is to  $EL$ , by VI. 4. Thus  $X$  is to  $Z$  in the duplicate ratio of  $X$  to  $Y$ . Therefore  $Y$  is a mean proportional between  $X$  and  $Z$ .

408. Join  $EG$  cutting  $AF$  at  $P$ , and  $HK$  cutting  $FC$  at  $Q$ . The triangles  $AEF$  and  $FHC$  are equiangular; therefore  $AE$  is to  $AF$  as  $FH$  is to  $FC$ . But  $AP$  is half of  $AF$ , and  $FQ$  is half of  $FC$ ; therefore  $AE$  is to  $AP$  as  $FH$  is to  $FQ$ . Therefore the triangle  $AEP$  is equiangular to the triangle  $FHQ$ , by VI. 6, so that the angle  $APE$  is equal to the angle  $FQH$ . Therefore  $EP$  is parallel to  $HQ$ , by I. 28.

409. Let  $ABC$  be the triangle. From  $C$  draw  $CH$  perpendicular to  $AB$ , produced if necessary; and complete the rectangle  $AFCH$ . Describe a rectangle  $AEDG$  similar to  $AHCF$ , and equal to the triangle  $ABC$ , so that  $E$  may be on  $AH$  and  $G$  on  $AF$ , by VI. 25. Then  $D$  will fall on  $AC$ , by VI. 26.

Then the triangle  $AED$  is half the rectangle  $AEDG$ , and is therefore equal to half the triangle  $ABC$ . Thus the straight line  $ED$  satisfies the assigned conditions.

410. Let  $ABC$ ,  $DEF$  be the two isosceles triangles which are to one another in the duplicate ratio of their bases  $BC$ ,  $EF$ . Then  $ABC$  and  $DEF$  shall be similar triangles.

For if the angle  $ABC$  be not equal to the angle  $DEF$ , one of them must be the greater. Let  $ABC$  be the greater, and make the angle  $CBG$  equal to the angle  $FED$ . Similarly make the angle  $BCG$  equal to the angle  $EFD$ . Then  $GBC$  and  $ABC$  being isosceles triangles the point  $G$  will fall within the angle  $BAC$ , so that the triangle  $ABC$  is greater than the triangle  $GBC$ . Also  $DEF$  and  $GBC$  are similar triangles.

Now, by supposition, the triangle  $DEF$  is to the triangle  $ABC$  in the duplicate ratio of  $EF$  to  $BC$ ; and the triangle  $DEF$  is, by VI. 19, to the triangle  $GBC$  in the same ratio; therefore the triangle  $DEF$  is to the triangle  $ABC$  as the triangle  $DEF$  is to the triangle  $GBC$ . Therefore the triangle  $ABC$  is equal to the triangle  $GBC$ , which is absurd. Therefore the triangle  $ABC$  is similar to the triangle  $DEF$ .

411. The rectangle contained by the two segments is known, for it is equal to that of the segments of any chord of the circle through the point. Also the ratio of the sides of the rectangle is known. Hence the rectangle can be constructed by VI. 25.

412. The rectangle contained by the line and one segment is known, for it is equal to the square on the tangent. Also one side of the rectangle is to be double the other. Hence the rectangle can be constructed by VI. 25.

413. The straight line  $CD$  is divided similarly to  $AB$ . The straight line  $EB$  is divided similarly to  $AB$ : see VI. 2. Also it is shewn in II. 11 that the rectangle  $CF$ ,  $FA$  is equal to the square on  $AC$ ; therefore  $CF$  is divided at  $A$  in the required manner. And  $KG$  is divided at  $H$  similarly to the way in which  $CF$  is divided at  $A$ .

414. Let  $BC$  denote the given base; on  $BC$  describe a segment of a circle containing an angle equal to the given vertical angle. Then the diameter of this circle is known.

By VI. C the perpendicular from the vertex on the base of the triangle is known. Hence we must draw a straight line parallel to  $BC$  at a distance from it equal to this known perpendicular; either of the intersections of this straight line with the arc of the described segment may be taken for the required vertex of the triangle.

415. Let  $ABC$  be an equilateral triangle; suppose a circle to be described round the triangle, and let  $P$  be any point on the circumference of this circle. Draw  $PA$ ,  $PB$ ,  $PC$ .

Suppose  $P$  to be between  $A$  and  $C$ . Then  $APCB$  is a quadrilateral inscribed in a circle; and therefore the rectangle  $PB$ ,  $AC$  is equal to the sum of the rectangles  $PA$ ,  $BC$  and  $PC$ ,  $AB$ , by VI. D. But  $AC$ ,  $BC$ , and  $AB$  are all equal, by supposition. Therefore  $PB$  is equal to the sum of  $PA$  and  $PC$ .

416. Since the angles  $ABD$  and  $ACD$  are right angles a circle would go round  $ABDC$ , by page 276 of the *Euclid*; therefore the rectangle  $AD$ ,  $BC$  is

equal to the sum of the rectangles  $AB, CD$  and  $AC, BD$ , by VI. D. But  $AB$  is equal to  $AC$ , therefore the angle  $ABC$  is equal to the angle  $ACB$ , by I. 5; therefore the angle  $DBC$  is equal to the angle  $DCB$ ; therefore  $DB$  is equal to  $DC$ , by I. 6. Thus each of the rectangles  $AB, CD$  and  $AC, BD$  is equal to the rectangle  $AB, DB$ . Therefore the rectangle  $AD, BC$  is equal to twice the rectangle  $AB, DB$ .

417. Suppose that the straight line through  $A$  falls without the triangle  $ABC$ , and that the perpendicular  $CE$  is less than  $FG$ .

The angle  $AFC$  is a right angle by I. 8; therefore a circle would go round  $AECF$ , by page 276 of the *Euclid*. Therefore the rectangle  $AC, EF$  is equal to the sum of the rectangles  $AE, FC$  and  $AF, EC$ . From  $F$  draw  $FH$  perpendicular to  $EC$  produced; then  $EC$  is the excess of  $EH$  over  $HC$ , that is equal to the excess of  $FG$  over  $HC$ . Therefore the rectangle  $AC, EF$  is equal to the excess of the rectangles  $AE, FC$  and  $AF, FG$  over the rectangle  $AF, HC$ . Now the triangle  $AFG$  is equiangular to the triangle  $HFC$ , for  $AF$  and  $FH$  are respectively perpendicular to  $FC$  and  $FG$ ; so that  $FA$  is to  $AG$  as  $FC$  is to  $HC$ , by VI. 4; and therefore the rectangle  $FA, HC$  is equal to the rectangle  $FC, AG$ .

Therefore the rectangle  $AC, EF$  is equal to the excess of the rectangles  $AE, FC$  and  $AF, FG$  over the rectangle  $FC, AG$ , that is equal to the rectangles  $AF, FG$  and  $FC, EG$ .

The demonstration will remain substantially the same for other forms of the diagram.

#### XI. 1 to 12.

418. Let  $PA, PB$  be two equal straight lines drawn from a point  $P$  to a plane. Let  $PN$  be perpendicular to the plane.

The angles  $PNA$  and  $PNB$  are right angles; therefore the square on  $PA$  is equal to the sum of the squares on  $PN$  and  $NA$ , and the square on  $PB$  is equal to the sum of the squares on  $PN$  and  $NB$ . But  $PA$  is equal to  $PB$ ; therefore  $NA$  is equal to  $NB$ . Therefore the angle  $PAN$  is equal to the angle  $PBN$ , by I. 8.

419. Suppose  $AB, AC$  to be two straight lines in one plane equally inclined to another plane; and let the planes intersect in  $BC$ . From  $A$  draw  $AD$  perpendicular to the second plane. Then the angle  $ABD$  is equal to the angle  $ACD$  by supposition; the right angle  $ADB$  is equal to the right angle  $ADC$ ; and  $AD$  is common to the two triangles  $ADB$  and  $ADC$ . Therefore  $AB$  is equal to  $AC$ , by I. 26; and therefore the angle  $ABC$  is equal to the angle  $ACB$ , by I. 5.

If the point  $A$  is in the line of intersection of the two planes this method does not apply. Then take  $AB$  equal to  $AC$ ; from  $B$  and  $C$  draw  $BF, CG$  perpendicular to the line of intersection of the planes, and from  $F$  and  $G$  draw  $FD, GE$  in the second plane perpendicular to the line of intersection of the planes; and from  $B$  and  $C$  draw  $BD, CE$  perpendicular respectively to the straight lines  $FD$  and  $GE$ . Then  $BD, CE$  will be perpendicular to the second plane by the construction in XI. 11; and the angles  $BFD$ , and  $CGE$  will each measure the inclination of the two given planes by XI. Def. 6. Join  $AD, AE$ . Then  $AB$  is equal to  $AC$ ; the angle  $BAD$  is equal to the angle  $CAE$ , by supposition; and the right angle  $BDA$  is equal to the right angle  $CEA$ : therefore  $BD$  is equal to  $CE$ , by I. 26.

The angle  $BFD$  is equal to the angle  $CGE$ , for each measures the inclination of one plane to the other; therefore  $BF$  is equal to  $CG$ , by I. 26. Also  $AB$  is equal to  $AC$ ; therefore  $AF$  is equal to  $AG$ : see I. 47. Therefore the angle  $BAF$  is equal to the angle  $CAG$ , by I. 8.

420. Let  $DCE$  be the straight line in the plane. Through  $B$  draw in the plane a straight line  $FBG$  at right angles to  $BC$ . Then  $GBF$  is at right angles to both  $BC$  and  $BA$ , and is therefore at right angles to the plane  $ABC$ , by XI. 4. Now  $DE$  and  $GF$  are both in one plane, and both at right angles to  $BC$ ; therefore  $DE$  is parallel to  $GF$ , by I. 28. Therefore  $DE$  is at right angles to the plane  $ACB$ , by XI. 8. Therefore  $DE$  is at right angles to  $AC$ ; that is  $AC$  is at right angles to  $DE$ .

421. In the plane  $ABC$  draw  $GCF$  parallel to  $AB$ ; produce  $CD$  to meet  $AB$  at  $K$ ; and from  $K$  draw  $KL$  parallel to  $DE$ .

Then  $DE$  is perpendicular to the plane  $ABC$  by supposition; therefore  $LK$  is perpendicular to this plane, by XI. 8; therefore  $BKL$  is a right angle. And  $BKC$  is a right angle, see the *Euclid*, p. 813. Therefore  $BK$  is perpendicular to the plane containing  $LK$ ,  $DE$ , and  $KC$ . Therefore  $GCF$  is perpendicular to this plane, by XI. 8. Therefore  $GC$  is at right angles to  $CE$ .

422. Let  $A$  and  $B$  be the two given points. From  $A$  draw  $AC$  perpendicular to the plane, and produce  $AC$  to  $D$  making  $CD$  equal to  $CA$ . Join  $DB$  intersecting the given plane at  $P$ . Then  $AP$  and  $BP$  are together less than any two other straight lines which can be drawn from  $A$  and  $B$  to meet in the plane.

For take any point  $Q$  in the plane; join  $AQ$  and  $DQ$ . Then in the two triangles  $ACQ$  and  $DCQ$  the sides  $AC$  and  $DC$  are equal; the side  $CQ$  is common; and the right angle  $ACQ$  is equal to the right angle  $DCQ$ : therefore  $AQ$  is equal to  $DQ$ , by I. 4. In the same manner it can be shewn that  $AP$  is equal to  $DP$ . Therefore the sum of  $AQ$  and  $BQ$  is equal to the sum of  $DQ$  and  $BQ$ ; and is therefore greater than  $DB$ , by I. 20. But  $DB$  is the sum of  $DP$  and  $PB$ , and is therefore equal to the sum of  $AP$  and  $BP$ . Thus the sum of  $AP$  and  $BP$  is less than the sum of  $AQ$  and  $BQ$ .

423. Let  $OA$ ,  $OB$ ,  $OC$  be the three equal straight lines, meeting at  $O$ . From  $O$  draw  $OP$  perpendicular to the plane  $ABC$ ; join  $PA$ ,  $PB$ ,  $PC$ .

The angles  $OPA$ ,  $OPB$ ,  $OPC$  are right angles. Hence by I. 47 it can be shewn that  $PA$ ,  $PB$ ,  $PC$  are all equal; so that  $P$  is the centre of the circle described round the triangle  $ABC$ .

424. Let the three straight lines meet at  $O$ . Take on these straight lines equal lengths  $OA$ ,  $OB$ ,  $OC$ . From  $O$  draw  $OP$  perpendicular to the plane  $ABC$ . Then  $OP$  is the required straight line.

For  $PA$ ,  $PB$ ,  $PC$  are all equal by Exercise 423; therefore the angles  $POA$ ,  $POB$ ,  $POC$  are all equal, by I. 8.

425. Since  $EC$ ,  $DF$  are perpendicular to the same plane they are parallel, by XI. 6: therefore the points  $E$ ,  $C$ ,  $D$ ,  $F$  are in one plane.

Let  $CF$ , produced if necessary, meet  $AB$  at  $G$ . Draw the straight line  $GH$  parallel to  $EC$  or  $DF$ . Then  $GH$  is at right angles to the plane  $CAB$ , by XI. 8; and therefore the angle  $AGH$  is a right angle. Similarly a straight line  $GK$  drawn parallel to  $ED$  will lie in the plane  $ECD$ , and will be at right angles to the plane  $DAB$ ; therefore the angle  $AGK$  is a right angle. Thus

the straight line  $AB$  will be at right angles to the plane in which  $GH$  and  $GK$  lie, by XI. 4; and will therefore be at right angles to the straight line  $CFG$  which lies in that plane.

426. From a point  $O$  draw a straight line  $OP$  perpendicular to a given plane, and a straight line  $OQ$  perpendicular to the straight line  $AB$  lying in that plane. Then  $PQ$  will be perpendicular to  $AB$ .

From  $Q$  draw  $QL$  parallel to  $OP$ ; then  $QL$  is at right angles to the plane, by XI. 8; so that the angle  $LQA$  is a right angle. Also  $AQO$  is a right angle, by supposition. Therefore  $AQ$  is at right angles to the plane containing  $OQ$  and  $LQ$ , by XI. 4: so that the angle  $AQP$  is a right angle.

#### XI. 13 to 21.

427.  $AB$  is perpendicular to the plane  $BED$ , and  $AC$  is perpendicular to the plane  $CED$ ; therefore the straight line  $ED$  is perpendicular to the plane  $ABCE$ , by XI. 18, 19. Therefore the angles  $DEB$  and  $DEC$  are right angles. Since  $ABE$  and  $ACE$  are right angles a circle would pass round  $ACEB$ ; and therefore  $CAB$  and  $CEB$  are together equal to two right angles: see page 276 of the *Euclid*. Thus the four angles  $CAB$ ,  $CEB$ ,  $DEB$ ,  $DEC$  are together equal to four right angles.

428. Suppose  $ABC$  to denote a triangle, and  $KLM$  another triangle lying within the former: then it may be shewn that the *perimeter* of the second triangle is less than the *perimeter* of the former. The demonstration will be a series of steps of the following kind: produce  $KL$  to cut a side of the triangle  $ABC$ , say to cut the side  $AC$  at  $P$ : then  $LP$  and  $PM$  together are greater than  $LM$ , by I. 20; therefore the perimeter of  $KPM$  is greater than the perimeter of  $KLM$ . In this way we finally obtain the required result by repeated application of I. 20.

Now in the problem at present under consideration we must use XI. 20 instead of I. 20. Thus let  $O$  denote the point not in the plane of the triangles: then the sum of the angles  $LOP$  and  $POM$  is greater than the angle  $LOM$ ; therefore the sum of the angles subtended at  $O$  by the sides of  $KPM$  is greater than the sum of the angles subtended at  $O$  by the sides of  $KLM$ . In this way we finally obtain the required result by repeated application of XI. 20.

429. Draw  $AE$  parallel to  $ab$ , meeting  $Bb$ , produced if necessary, at  $E$ ; draw  $CF$  parallel to  $cd$ , meeting  $Dd$ , produced if necessary, at  $F$ . Then  $ab$  and  $cd$  are parallel, by XI. 15 and XI. 16. Therefore  $AE$  and  $CF$  will be parallel, by XI. 9. Therefore the triangle  $AEB$  is equiangular to the triangle  $CFD$ ; so that  $AB$  is to  $CD$  as  $AE$  is to  $CF$ . But  $AE$  is equal to  $ab$ , and  $CF$  is equal to  $cd$ , by I. 34. Therefore  $AB$  is to  $CD$  as  $ab$  is to  $cd$ .

430. Let  $ABCD$  be the regular tetrahedron. From  $A$  draw  $AF$  perpendicular to the plane  $BCD$ . Then  $F$  is the centre of the circle which would go round  $BCD$ ; so that  $FB$ ,  $FC$ ,  $FD$  are all equal: see Exercise 423. Produce  $BF$  to meet  $CD$  at  $G$ ; then it may be shewn that  $BG$  is at right angles to  $CD$ . Also  $FG$  will be one third of  $BG$ : see Exercise 343.

From  $F$  draw  $FK$  perpendicular to  $AG$ , and from  $B$  draw  $BL$  perpendicular to  $AG$ . Then the triangle  $BGL$  is equiangular to the triangle  $FKG$ ; and  $BL$  is three times  $FK$ , since  $BG$  is three times  $FG$ , by VI. 4. Now  $FK$  is



perpendicular to the plane  $ACD$ , by XI. 11; and so also is  $BL$ . Thus the perpendicular from  $B$  on the face  $ACD$  is three times the perpendicular from  $F$  on the face. And the perpendicular from  $B$  on the face  $ACD$  is equal to the perpendicular from  $A$  on the face  $BCD$  by reason of the symmetry of the regular tetrahedron.

431. Since the angles at the vertex are right angles each of the three faces meeting at the vertex is at right angles to the other two. Let  $BAC$  be the equilateral base, and  $D$  the vertex. Then since  $BA$  is equal to  $BC$ , and  $BD$  is common to the triangles  $BDA$  and  $BDC$ , we have  $AD$  equal to  $CD$ , by I. 47. Similarly  $AD$  is equal to  $BD$ .

Now let  $P$  be any point of the base. From  $P$  draw  $Pm$  perpendicular to the plane  $ADC$ , and  $Pn$  perpendicular to the plane  $ADB$ . Then the plane  $mPn$  will by XI. 18 be perpendicular to both the planes  $ADC$ ,  $ADB$ , and therefore also to  $AD$  their line of intersection. Also the plane  $mPn$  is parallel to  $BDC$  by XI. 14; let it cut  $AC$  in  $c$ ,  $AB$  in  $b$ ,  $AD$  in  $a$ . Then  $ba$  is parallel to  $BD$ , and  $ca$  is parallel to  $CD$  by XI. 16. Also  $aD$  is equal to the perpendicular from  $P$  on the plane  $BDC$ .

The triangle  $Pmc$  is equiangular to the triangle  $BDC$ ; therefore  $Pm$  is equal to  $mc$ , by VI. 4. Thus the sum of  $Pm$  and  $Pn$  is equal to the sum of  $mc$  and  $ma$ ; that is equal to  $ac$ ; that is equal to  $aA$ , since  $ac$  is parallel to  $DC$ . Therefore the sum of  $Pm$ ,  $Pn$ , and  $aD$  is equal to  $AD$ , and is therefore constant.

432. Let  $OA$ ,  $OB$ ,  $OC$  be the three straight lines which meet at  $O$ . Through  $O$  draw any straight line  $OQ$  meeting the plane  $ABC$  at a point  $Q$  within the triangle  $ABC$ .

Then by the aid of I. 20 we can shew that the sum of  $QA$ ,  $QB$ , and  $QC$  is less than the sum of  $AB$ ,  $BC$ , and  $CA$ , but greater than half the sum of  $AB$ ,  $BC$ , and  $CA$ : see Exercises 22 and 441. Now in precisely the same manner the present Exercise may be established, using XI. 20 instead of I. 20; and instead of a straight line, as  $QA$ , the corresponding angle  $QOA$ .

433. Let one plane cut the three straight lines at  $A$ ,  $B$ ,  $C$  respectively; let another plane, parallel to this, cut the straight lines at  $K$ ,  $L$ ,  $M$ , respectively; let a third plane cut the straight lines at  $P$ ,  $Q$ ,  $R$ , respectively, where  $P$ ,  $Q$ ,  $R$  are not in a straight line. And suppose  $AKP$ ,  $BLQ$ ,  $CMR$  to be cut in the same ratio by the planes. Then the plane  $PQR$  will be parallel to the other two planes.

For if not draw a plane through  $PQ$  parallel to the planes  $ABC$  and  $KLM$ ; and let this cut the straight line  $CMR$  at  $S$ .

Then  $CR$  is to  $RM$  as  $BQ$  is to  $QL$ , by supposition; and  $CS$  is to  $SM$  as  $BQ$  is to  $QL$ , by XI. 17: therefore  $CR$  is to  $RM$  as  $CS$  is to  $SM$ ; which is impossible. The condition that  $P$ ,  $Q$ ,  $R$  should not be on one straight line is necessary; for otherwise an indefinite number of planes could pass through  $P$ ,  $Q$ ,  $R$ .

434. Let  $AB$  and  $CD$  be the two straight lines. At the point  $A$  draw  $AK$  parallel to  $CD$ ; and at the point  $C$  draw  $CL$  parallel to  $AB$ . Thus the planes  $BAK$ ,  $LCD$  are the planes required.

For these planes are parallel, by XI. 15; and the former plane passes through the straight line  $AB$ , and the latter through the straight line  $CD$ .

435. Let the planes  $ABCD$ ,  $KLMN$  be parallel. Let a plane cut  $ABCD$  in  $OP$ , and  $KLMN$  in  $RS$ . Let another plane cut  $ABCD$  in  $OQ$ , and  $KLMN$  in  $RT$ . Then the angle  $POQ$  will be equal to the angle  $SRT$ .

For  $RS$  is parallel to  $OP$ , and  $RT$  is parallel to  $OQ$ , by XI. 16; therefore the angle  $POQ$  is equal to the angle  $SRT$ , by XI. 10.

436. The plane  $ABC$  is at right angles to both planes, by XI. 18; and therefore to their common intersection, by XI. 19. Therefore any straight line in the plane  $ABC$  is perpendicular to their common intersection; and  $BC$  is such a straight line; therefore  $BC$  is perpendicular to their common intersection.

437. Let  $AB$  and  $BC$  be two consecutive sides of a polygon obtained by cutting a prism by a plane; let  $ab$  and  $bc$  be the corresponding sides of the polygon obtained by cutting the prism by a plane parallel to the former. Then  $Aa$  and  $Bb$  are parallel by the definition of a prism; and  $AB$  and  $ab$  are parallel, by XI. 16; therefore  $ABba$  is a parallelogram; therefore  $AB$  is equal to  $ab$ , by I. 34. Similarly  $BC$  is equal to  $bc$ ; and so on.

438. Let  $AB$  and  $BC$  be two consecutive sides of a polygon obtained by cutting a pyramid by a plane; let  $ab$  and  $bc$  be the corresponding sides of the polygon obtained by cutting the pyramid by a plane parallel to the former. Let  $O$  be the vertex of the pyramid.

Then  $AB$  is parallel to  $ab$ , and  $BC$  is parallel to  $bc$ , by XI. 16; therefore  $AB$  is to  $ab$  as  $OB$  is to  $Ob$ , by VI. 4. Similarly  $BC$  is to  $bc$  as  $OB$  is to  $Ob$ . Therefore  $AB$  is to  $ab$  as  $BC$  is to  $bc$ . Also the angle  $ABC$  is equal to the angle  $abc$ , by XI. 10. Since these results hold for any corresponding pair of consecutive sides the polygons are similar.

439. The angle  $ABC$  is equal to the angle  $abc$ , by XI. 10. Also  $AB$  is to  $ab$  as  $PB$  is to  $Pb$ , by VI. 4; that is as  $pb$  is to  $pB$ , by supposition; that is as  $bc$  is to  $BC$ , by VI. 4. Therefore the triangle  $ABC$  is equal to the triangle  $abc$ , by VI. 15.

440. Let  $OC$  be the line of intersection, where  $O$  is in  $EF$ , or  $EF$  produced. The plane  $AOE$  contains  $AB$  and is therefore perpendicular to the plane  $AOC$ , by XI. 18; similarly the plane  $AOE$  contains  $AE$  and is therefore perpendicular to the plane  $EOC$ . Therefore  $OC$  is perpendicular to the plane  $AOE$ , by XI. 19; therefore  $OC$  is perpendicular to the straight line  $EF$  in that plane.

#### I. 1 to 48.

441. The sum of  $BP$  and  $CP$  is less than the sum of  $BA$  and  $CA$ , by I. 21; similarly the sum of  $CP$  and  $AP$  is less than the sum of  $CB$  and  $AB$ ; and the sum of  $AP$  and  $BP$  is less than the sum of  $AC$  and  $BC$ . Thus twice the sum of  $AP$ ,  $BP$ , and  $CP$  is less than twice the sum of  $AC$ ,  $BA$ , and  $CB$ ; and therefore the sum of  $AP$ ,  $BP$ , and  $CP$  is less than the sum of  $AC$ ,  $BA$ , and  $CB$ .

442. The angle  $APR$  is equal to the angle  $BQS$ , by I. 29. The angle  $APR$  is equal to the angle  $ARP$ , and the angle  $BQS$  is equal to the angle  $BSQ$ , by I. 5; therefore the angle  $ARP$  is equal to the angle  $BSQ$ . Therefore  $AR$  is parallel to  $BS$ , by I. 28.

443. Let  $ABCD$  be a parallelogram, and  $P$  any point within it. The triangles  $PAB$  and  $PDC$  will be together half the parallelogram.

Through  $P$  draw a straight line parallel to  $AB$  and  $DC$ , meeting  $AD$  at  $K$ , and  $BC$  at  $M$ . The triangle  $PDC$  is half the parallelogram  $KDCM$ , by I. 41; and also the triangle  $PAB$  is half the parallelogram  $KABM$ . Therefore the triangles  $PDC$  and  $PAB$  together are equal to half the sum of the parallelograms  $KDCM$  and  $KABM$ ; that is to half the parallelogram  $ABCD$ .

444. Let  $ABCD$  be a quadrilateral, such that the diagonal  $AC$  bisects it. Then the triangle  $ADC$  is equal to the triangle  $ABC$ : therefore  $BD$  is bisected by  $AC$ , by Exercise 114.

445. Let  $ABCD$  be the quadrilateral figure. The triangle  $ABC$  is equal to the triangle  $ABD$ ; for each is half the quadrilateral by supposition. Therefore  $DC$  is parallel to  $AB$ , by I. 39.

Similarly  $BC$  is parallel to  $AD$ .

446. Produce  $CA$  to  $M$ , and  $BA$  to  $N$ , making  $CM$  equal to  $BN$ .

In the triangles  $MCB$  and  $NBC$  the side  $CB$  is common; the side  $MC$  is equal to the side  $NB$ , by construction; and the angle  $MCB$  is equal to the angle  $NBC$ , by I. 5. Therefore the side  $MB$  is equal to the side  $NC$ , and the angle  $BMC$  is equal to the angle  $CNB$ .

In the triangles  $BMA$  and  $CNA$  we have the side  $MB$  equal to the side  $NC$ , and the angle  $BMA$  equal to the angle  $CNA$ , as just shewn; also the side  $MA$  equal to the side  $NA$ ; therefore the angle  $MAB$  is equal to the angle  $NAC$ , by I. 4.

447. From the given straight line cut off  $BC$  equal to the given length. Join  $AC$ ; draw from  $A$  a straight line  $AP$  meeting  $CB$  at  $P$ , and making the angle  $CAP$  equal to the angle  $ACP$ . Suppose that  $P$  falls between  $B$  and  $C$ .

Then  $AP$  is equal to  $CP$ , by I. 6; and therefore the sum of  $AP$  and  $PB$  is equal to  $CB$ .

If  $P$  does not fall between  $B$  and  $C$  the problem is impossible. But then the difference between  $AP$  and  $PB$  may be made equal to the given length.

448. In the first case of I. 26 suppose the triangle  $DEF$  applied to the triangle  $ABC$ , so that  $EF$  may be on  $BC$ , and the triangle  $DEF$  on the same side of  $BC$  as the triangle  $ABC$  is. Then since the angle  $DFE$  is equal to the angle  $ACB$ , and the angle  $DEF$  is equal to the angle  $ABC$ , the triangle  $DEF$  will coincide with the triangle  $ABC$ , and therefore be equal to it.

In the second case of I. 26 suppose the triangle  $DEF$  applied to the triangle  $ABC$ , so that  $DE$  may be on  $AB$ , and the triangle  $DEF$  on the same side of  $AB$  as the triangle  $ABC$ . Then since the angle  $DEF$  is equal to the angle  $ABC$  the straight line  $EF$  will fall on  $BC$ . Also  $DF$  will fall on  $AC$ ; for if not let it take a different position as  $AH$ . Then the angle  $AHB$  coincides with the angle  $DFE$  and is equal to it; but the angle  $DFE$  is equal to the angle  $ACB$ , by supposition; therefore the angle  $AHB$  is equal to the angle  $ACB$ . But this is impossible, by I. 16. Therefore  $DF$  cannot fall otherwise than on  $AC$ . Therefore the triangle  $DEF$  coincides with the triangle  $ABC$ , and is equal to it.

449. Let  $ABC$  be a triangle having the sides  $AB$  and  $AC$  equal. Let a straight line  $EDF$  meet  $AB$  at  $E$ , meet  $BC$  at  $D$ , and meet  $AC$  produced at

$F$ ; and let  $EF$  be bisected at  $D$ : then  $AE$  and  $AF$  together will be equal to  $AB$  and  $AC$  together.

From  $F$  draw  $FG$  parallel to  $AB$ , meeting  $BC$  produced at  $G$ . In the triangles  $EDB$  and  $FDG$  the sides  $ED$  and  $FD$  are equal, by supposition; the angles  $EDB$  and  $FDG$  are equal, by I. 15; and the angles  $EBD$  and  $FGD$  are equal by I. 29: therefore  $FG$  is equal to  $EB$ , by I. 26.

Also the angle  $FCG$  is equal to the angle  $ACB$ , by I. 15; and is therefore equal to the angle  $EBD$ , by I. 5. Thus the angle  $FGC$  is equal to the angle  $FCG$ ; and therefore  $FG$  is equal to  $FC$ , by I. 6. Thus  $FC$  is equal to  $EB$ ; therefore the sum of  $AB$  and  $AC$  is equal to the sum of  $AE$  and  $AF$ .

450. Suppose the straight line  $DME$  to meet  $AB$  at  $D$ , and to meet  $AC$  produced at  $E$ ; and let  $AD$  be equal to  $AE$ : then will  $BD$  be equal to  $CE$ .

Through  $C$  draw  $CF$  parallel to  $AB$ , meeting  $DE$  at  $F$ . In the triangles  $DMB$  and  $MFC$  the sides  $BM$  and  $CM$  are equal, by supposition; the angles  $DMB$  and  $FMC$  are equal, by I. 15; and the angles  $DBM$  and  $MCF$  are equal by I. 29: therefore  $DB$  is equal to  $FC$ , by I. 26.

The angle  $CFE$  is equal to the angle  $ADE$ , by I. 29; the angle  $ADE$  is equal to the angle  $AED$ , by I. 5: therefore the angle  $CFE$  is equal to the angle  $CEF$ . Therefore  $CF$  is equal to  $CE$ , by I. 6. Thus  $CE$  is equal to  $BD$ .

451. Let  $AB$  be one of the diagonals, and the triangle  $ABC$  half of one of the parallelograms. Let  $O$  be the middle point of  $AB$  and join  $OC$ . Then  $OC$  is half of the other diagonal, by Exercise 78. Thus  $AB$  and  $OC$  are given; and it is manifest that the triangle  $ABC$  has its greatest possible value when  $OC$  is at right angles to  $AB$ ; and then  $AC$  is equal to  $BC$ , by I. 4. Similarly any other two adjacent sides must be equal, so that the parallelogram will be a rhombus.

452. If  $AD$  be not equal to  $BD$  and  $CD$  it must be either greater or less than them. If possible suppose  $AD$  greater. Then the angles  $ABD$  and  $ACD$  are together greater than  $BAD$  and  $CAD$  together, by I. 18; therefore the three angles of the triangle  $ABC$  are greater than two right angles; but this is impossible, by I. 32. Therefore  $AD$  cannot be greater than  $BD$ .

Similarly it may be shewn that  $AD$  cannot be less than  $BD$ .

453. Let  $AB$  and  $CD$  be two equal straight lines which intersect at right angles. The quadrilateral  $ACBD$  will be equal to half the square on  $AB$  or  $CD$ .

Through  $C$  and  $D$  draw straight lines parallel to  $AB$ , and through  $A$  and  $B$  draw straight lines parallel to  $CD$ ; thus a parallelogram is formed; and it has all its sides equal, and all its angles right angles, so that it is a square. Also each triangle  $ABC$ ,  $ABD$  is half the corresponding rectangle of which  $AB$  is a side. Therefore the sum of the two triangles is half the sum of the two rectangles, that is half the square on  $AB$ .

454. Let  $ABC$  be the given triangle,  $D$  the given point within it. Draw  $AD$  and produce it to meet  $BC$  at  $E$ . On  $DA$  take  $DF$  equal to  $DE$ ; and through  $F$  draw a straight line parallel to  $BC$ , meeting  $AB$  at  $G$ , and  $AC$  at  $H$ . Produce  $GD$  to meet  $BC$  at  $K$ , and produce  $HD$  to meet  $BC$  at  $L$ ; then will  $GHLK$  be a parallelogram.

In the triangles  $GDF$  and  $KDE$  the sides  $DF$  and  $DE$  are equal, by construction; the angles  $GDF$  and  $KDE$  are equal, by I. 15; and the angles

$FGD$  and  $EKD$  are equal, by I. 29. Therefore  $GF$  is equal to  $KE$ , by I. 26. In like manner by comparing the triangles  $HDF$  and  $LDE$  we find that  $HF$  is equal to  $LE$ . Thus  $GH$  is equal to  $LK$ ; and therefore  $GL$  is equal and parallel to  $HK$ , by I. 38.

Instead of joining the point  $A$  with  $D$  we might join  $B$  or  $C$  with  $D$ . Thus three solutions occur; but we must have  $AD$  greater than  $DE$  in order that the first solution may be possible; and similar conditions hold with respect to the other two solutions.

455. Let  $AB$  be one of the given sides. On  $AB$  describe a triangle  $ABD$  having its area equal to the given area. Through  $D$  draw a straight line  $DE$  parallel to  $AB$ . With centre  $A$  and radius equal to the other given side describe a circle cutting  $DE$  at  $C$ . Then  $ACB$  will be the triangle required.

For the sides  $AB$  and  $AC$  have the prescribed lengths; and the area of  $ABC$  is equal to the area of  $ABD$ , by I. 37, and therefore has the prescribed value.

456. Let  $AB$  be the given base. At the point  $B$  make the angle  $ABD$  equal to half the difference of the angles at the base. With centre  $A$  and radius equal to the given difference of the sides describe a circle meeting  $BD$  at  $D$  and  $E$ , and let  $E$  be the nearer of the two points to  $B$ . Produce  $AE$  to any point  $F$ ; at the point  $B$  make the angle  $EBG$  equal to the angle  $FEB$ ; and let  $BG$  meet  $EF$  at  $C$ . Then  $ABC$  will be the triangle required.

For  $EC$  is equal to  $BC$ , by I. 5; therefore the difference of  $AC$  and  $BC$  is equal to  $AE$ . The angle  $ABC$  is the sum of the angles  $EBC$  and  $ABE$ ; the angle  $CAB$  is the difference of the angles  $CEB$  and  $ABE$ , by I. 32. Thus the angle  $ABC$  exceeds the angle  $CAB$  by twice the angle  $ABE$ , that is by the prescribed excess. And  $AB$  is the given base.

457. On  $AB$  take  $AF$  equal to the given straight line. Bisect the angle  $BAC$  by the straight line  $AE$ . From  $F$  draw a straight line at right angles to  $AB$ , meeting  $AE$  at  $G$ . From  $G$  draw a straight line meeting  $AB$  at  $P$ , making the angle  $AGP$  equal to the angle  $GAP$ . From  $P$  draw  $PQ$  perpendicular to  $AC$ .

The angle  $AGP$  is equal to the angle  $GAP$  by construction; therefore  $AP$  is equal to  $GP$ , by I. 5. The angle  $GPF$  is equal to the sum of the angles  $AGP$  and  $GAP$ , by I. 32; that is to twice the angle  $GAP$ ; that is to the angle  $PAQ$ . The angles  $PFQ$  and  $AQP$  are equal being right angles. Therefore  $PF$  is equal to  $AQ$ , by I. 26. Therefore the sum of  $AQ$  and  $AP$ , is equal to  $AF$ ; that is to the prescribed sum.

458. Let  $BC$  be the base of a triangle,  $D$  the middle point of the base; and let the angle  $BAC$  of the triangle be a right angle; then  $AD$  is equal to  $BD$ ; see Exercise 59 or Exercise 452.

Next let  $BAC$  be the triangle, and suppose  $BEC$  an acute angle. From  $C$  draw  $CA$  perpendicular to  $BE$ ; then  $CA$  falls within the triangle  $CBE$ . The angle  $DAE$  is greater than a right angle, and the angle  $DEA$  is less than a right angle; therefore  $DE$  is greater than  $DA$ , by I. 19. But  $DA$  is equal to  $BD$ , by the first case; therefore  $DE$  is greater than  $BD$ .

Finally let  $BFC$  be the triangle, and suppose  $BFC$  an obtuse angle. From  $C$  draw  $CA$  perpendicular to  $BF$  produced; then  $CA$  falls without the triangle  $CBF$ . The angle  $AFD$  is greater than the angle  $FBD$ , by I. 32; and therefore greater than the angle  $BAD$ , by I. 5; therefore  $DF$  is less than  $DB$ .

459. Let  $ABCD$  be a square. Take on  $AB$  a point  $E$ , on  $BC$  the point  $F$ , on  $CD$  the point  $G$ , and on  $DA$  the point  $H$ , such that  $AE, BF, CG$ , and  $DH$  are all equal : then will  $EFGH$  be a square.

For we have  $EB, FC, GD$ , and  $HA$  all equal. Thus the triangles  $HAE, EBF, FCG$ , and  $GDH$  are all equal, by I. 4 : therefore the figure  $EFGH$  is equilateral. It is also rectangular. For the angle  $HEA$  is equal to the angle  $EFB$ ; therefore the angles  $HEA$  and  $FEB$  are together equal to the angles  $EFB$  and  $FEB$  together; that is to a right angle, by I. 32. Therefore the angle  $HEF$  is a right angle by I. 13. Similarly the other angles of the figure  $EFGH$  are right angles.

460. Let  $AB$  be the given base,  $P$  the point through which a side is to pass. Join  $AP$  and produce it to  $E$ , and cut off a part  $AD$  equal to the given difference of the sides. Join  $BD$ . At the point  $B$  make the angle  $DBF$  equal to the angle  $BDE$ ; let  $DE$  intersect  $BF$  at  $C$ . Then  $ABC$  is the required triangle.

For the angle  $BDC$  is equal to the angle  $DBC$ , by construction; therefore  $BC$  is equal to  $DC$ , by I. 6. But  $AC$  exceeds  $CD$  by  $AD$ ; therefore  $AC$  exceeds  $BC$  by  $AD$ . Thus  $AC$  exceeds  $BC$  by the prescribed length; and  $AC$  passes through the given point  $P$ .

461. From  $AB$  cut off  $AE$  equal to  $AC$ , and join  $ED$ . Thus the triangles  $AED$  and  $ACD$  are equal in all respects by I. 4. The angle  $BED$  is greater than the angle  $ADE$ , by I. 16; therefore the angle  $BED$  is greater than the angle  $ADC$ . The angle  $ADC$  is greater than the angle  $ABD$ , by I. 16; therefore the angle  $BED$  is greater than the angle  $EBD$ ; therefore  $BD$  is greater than  $DE$ , by I. 19. But  $DE$  is equal to  $DC$ ; therefore  $BD$  is greater than  $DC$ .

462. Let  $ABC$  be a triangle, having the angle  $BAC$  triple the angle  $ABC$ .

Make the angle  $BAE$  equal to the angle  $ABC$ , and let  $AE$  meet  $BC$  at  $D$ . Then the angle  $DAC$  is double the angle  $ABC$ . And the angle  $ADC$  is equal to the angles  $ABD$  and  $BAD$ , by I. 32; therefore the angle  $ADC$  is equal to twice the angle  $ABD$ ; therefore the angle  $ADC$  is equal to the angle  $DAC$ . Thus  $BAD$  and  $CAD$  are isosceles triangles.

463. Let  $ABC$  be a triangle, having the angle  $BAC$  equal to double the angle  $ABC$ .

From  $C$  as centre, with radius equal to  $BC$  describe a circle meeting  $BA$  produced at  $D$ . Then  $BCD$  is an isosceles triangle; and therefore the angle  $ADC$  is equal to the angle  $ABC$ . The angle  $BAC$  is equal to the two angles  $ADC$  and  $ACD$ , by I. 32; and it is also equal to twice the angle  $ABC$ , that is to twice the angle  $ADC$ . Therefore the angle  $ADC$  is equal to the angle  $ACD$ ; therefore  $AD$  is equal to  $AC$ , by I. 6. Thus  $BCD$  and  $CAD$  are isosceles triangles.

464. Let  $ABC$  be the triangle having  $AB$  equal to  $AC$ . Let  $D$  be the middle point of  $AB$ . Produce  $AB$  to  $E$  so that  $EB$  is equal to  $BA$ . Then  $CE$  will be equal to twice  $CD$ .

Produce  $CD$  to  $F$  making  $DF$  equal to  $CD$ ; and join  $BF$ . Then the two triangles  $ADC$  and  $BDF$  are equal in all respects by I. 4; so that  $FB$  is equal to  $CA$ , and the angle  $DBF$  equal to the angle  $DAC$ .

In the two triangles  $CBF$  and  $CBE$  the side  $CB$  is common; the side  $BF$  is equal to  $AC$ , that is to  $AB$ , that is to  $BE$ ; the angle  $FBC$  is equal to the angles  $FBA$  and  $ABC$ , that is to the angles  $DAC$  and  $ACB$ , that is to the angle  $EBC$ , by I. 32; therefore  $FC$  is equal to  $EC$ , by I. 4. But  $FC$  is equal to twice  $DC$ , by construction; therefore  $EC$  is equal to twice  $DC$ .

465. Suppose  $D$  and  $E$  of the preceding Exercise to denote the fixed points; then every point  $C$  is a point on the required locus, so that the locus is a circle having its centre at  $A$  and its radius equal to  $AB$ .

466. Let  $H$  be opposite  $C$  in the parallelogram  $DCPH$ , and let  $K$  be opposite  $C$  in the parallelogram  $GCEK$ . Draw  $AH$ ,  $HB$ ,  $BK$ , and  $KA$ . Then  $HD$  is equal and parallel to  $BG$ , and  $DA$  is equal and parallel to  $GK$ ; thus the angle  $HDA$  is equal to the angle  $BGK$ , and  $HA$  is equal and parallel to  $BK$ . Similarly  $HB$  is equal and parallel to  $AK$ . Therefore  $AHBK$  is a parallelogram; and its diagonals bisect each other by Exercise 78. But  $C$  is the middle point of  $AB$ , and therefore also of  $HK$ .

467. Through  $F$  draw a straight line parallel to  $BC$ , meeting  $AB$  at  $G$ , and join  $EG$ . From  $BA$  cut off  $BH$  equal to  $DF$ , and join  $EH$ . Then the triangle  $BHE$  is half the rectangle  $BE, BH$ ; and is therefore equal to half the rectangle  $BE, DF$ . We have then to shew that the triangles  $AEF$  and  $BEH$  are together equal to half the rectangle  $ABCD$ .

The triangle  $BEH$  is equal to the triangle  $EGA$ , by I. 38; therefore the triangles  $AEF$  and  $BEH$  together are equal to the figure  $EGAF$ ; and therefore equal to the two triangles  $EGF$ , and  $GFA$ ; that is to half the rectangle  $BCFG$  together with half the rectangle  $GFDA$ ; that is to half the rectangle  $ABCD$ .

468. Take the case in which  $D$  is without the triangle  $ABC$ , and  $BD$  between  $BA$  and  $BC$ ; also suppose  $F$  to be on  $BA$  produced and  $E$  on  $CA$  produced.

Then  $ED$  is equal to  $EC$ , as they are radii of the same circle; and similarly  $FD$  is equal to  $FB$ . Then  $ED$  and  $AF$  together are equal to  $EC$  and  $AF$  together; that is to  $EA, AF$ , and  $AC$  together. Again  $FD$  and  $EA$  together are equal to  $FB$  and  $EA$  together; that is to  $EA, FA$ , and  $AB$  together, that is to  $EA, AF$ , and  $AC$  together. Therefore  $ED$  and  $AF$  together are equal to  $FD$  and  $EA$  together.

Similarly the other cases which arise from modifications of the diagram may be treated. For instance, if  $D$  be within the triangle  $ABC$ , the point  $F$  on  $AB$ , and the point  $E$  on  $AC$ , we shall find that  $EA$  and  $ED$  together are equal to  $FA$  and  $FD$  together.

In all cases the sides of the quadrilateral taken in order are  $AE, ED, DF, FA$ ; and if one of the sides  $BA, CA$ , requires to be produced, the other also will have to be produced.

469. From  $AB$  cut off  $AE$  equal to the required length. From  $E$  draw  $EH$  perpendicular to  $AC$ . Bisect the angle  $HEB$  by the straight line  $EM$  meeting  $AC$  at  $M$ ; from  $M$  draw  $MP$  at right angles to  $AC$ , meeting  $AB$  at  $P$ ; then  $P$  will be the required point.

For the angle  $EMP$  is equal to the angle  $MEH$ , by I. 29; and the angle  $MEH$  is equal to the angle  $MEP$ , by construction; therefore the angle  $EMP$  is equal to the angle  $MEP$ ; therefore  $PE$  is equal to  $PM$  by I. 6. Thus the excess of  $AP$  above  $PM$  is equal to the excess of  $AP$  above  $EP$ , that is equal to  $AE$ : and therefore it has the prescribed value.

470. Let  $ABCDEF$  be the equiangular hexagon. Join  $AD$ .

The two angles  $FAD$  and  $ADE$  together are equal to the excess of four right angles above  $AFE$  and  $FED$  together; that is to the excess of four right angles above two of the angles of the hexagon. Similarly the two angles  $BAD$  and  $ADC$  together are equal to the excess of four right angles above two of the angles of the hexagon. Therefore the angles  $FAD$  and  $ADE$  together are equal to the angles  $BAD$  and  $ADC$  together.

Again the angles  $FAD$  and  $DAB$  together are equal to an angle of the hexagon, and are therefore equal to  $EDA$  and  $ADC$  together. Therefore the angles  $FAD$ ,  $ADE$ ,  $EDA$  and  $ADC$  together are equal to the angles  $BAD$ ,  $ADC$ ,  $FAD$ , and  $DAB$  together. Therefore the angle  $EDA$  is equal to the angle  $BAD$ . Therefore  $AB$  is parallel to  $ED$ , by I. 27.

Produce  $AF$  and  $DE$  to meet at  $G$ , and produce  $AB$  and  $DC$  to meet at  $H$ . Then  $AHGD$  is a parallelogram, by what has been shewn. Therefore  $GD$  is equal to  $AH$ ; that is  $GE$  and  $ED$  together are equal to  $AB$  and  $BH$  together. Now since the hexagon is equiangular each angle is equal to two-thirds of two right angles, by I. 32, Corollary 1; therefore each of the angles  $GFE$  and  $GEF$  is one-third of two right angles; therefore the angle  $FGE$  is also one-third of two right angles, by I. 32. Thus the triangle  $FGE$  is equiangular; and therefore equilateral, by I. 6. Therefore  $GE$  is equal to  $EF$ . Similarly  $BC$  is equal to  $BH$ . Therefore  $FE$  and  $ED$  together are equal to  $AB$  and  $BC$  together.

471. From  $D$  draw  $DK$  perpendicular to  $AB$  produced. Then  $DKAM$  is a rectangular parallelogram, by I. 28; and therefore  $DK$  is equal to  $MA$ , by I. 34.

The angles  $DBK$  and  $CBA$  are together equal to a right angle, by I. 13. The angles  $CBA$  and  $BCA$  are together equal to a right angle, by I. 32. Therefore the angle  $DBK$  is equal to the angle  $BCA$ . Also the right angle  $DKB$  is equal to the right angle  $BAC$ ; and the side  $DB$  is equal to the side  $BC$ . Therefore  $DK$  is equal to  $BA$ , by I. 26. Therefore  $MA$  is equal to  $BA$ .

Similarly  $NA$  is equal to  $AC$ .

472. Through  $P$  draw a straight line parallel to  $AC$ , meeting  $AB$  at  $Q$ . From  $QB$  cut off  $QM$  equal to  $AQ$ . Join  $MP$  and produce it to meet  $AC$  at  $N$ . Then  $AMN$  is the triangle required.

For draw any other straight line  $KPL$  through  $P$ , meeting  $AB$  at  $K$  and  $AC$  at  $L$ : then the triangle  $AKL$  will be greater than the triangle  $AMN$ . Suppose  $AK$  greater than  $AM$ . From  $M$  draw  $MR$  parallel to  $AC$  meeting  $KL$  at  $R$ . Then  $MP$  is equal to  $NP$ ; see Exercise 106; the angle  $MPR$  is equal to the angle  $NPL$ , by I. 15; and the angle  $PMR$  is equal to the angle  $PNL$ , by I. 29; therefore the triangle  $PNL$  is equal to the triangle  $PMR$ , by I. 26. Hence the triangle  $AKL$  exceeds the triangle  $AMN$  by the triangle  $KMR$ .

The proof is similar, when  $AK$  is less than  $AM$ .



473. Let a straight line parallel to the given straight line meet  $CA$  at  $L$  and  $CB$  at  $M$ . At  $C$  make the angle  $LCK$  equal to  $MLC$ , and let  $CK$  cut  $AB$  at  $D$ . Through  $D$  draw a straight line parallel to  $LM$  meeting  $CA$  at  $E$  and  $CB$  at  $F$ . Then will  $EF$  be bisected at  $D$ .

For the angle  $DEC$  is equal to the angle  $MLC$ , by I. 29; and the angle  $DCE$  is equal to the angle  $MLC$  by construction; therefore the angle  $DCE$  is equal to the angle  $DEC$ . The angles  $DCE$  and  $DCF$  are together equal to a right angle; the angles  $DEC$  and  $DFC$  are together equal to a right angle, by I. 32; therefore the angle  $DCF$  is equal to the angle  $DFC$ . Therefore  $DE$  and  $DF$  are each equal to  $DC$ , by I. 6; therefore  $DE$  is equal to  $DF$ , so that  $FE$  is bisected at  $D$ .

474. Bisect  $BD$  at  $E$ . Then  $BE$  is equal to  $BC$ . The angle  $ABC$  is equal to the angles  $BEC$  and  $BCE$ , by I. 32. Therefore the angle  $ABC$  is twice the angle  $BEC$ . Therefore the angle  $BEC$  is half the angle  $ABC$ , and is therefore equal to twice the angle  $BAC$ , by supposition. And the angle  $EBC$  is equal to the angles  $BAC$  and  $BCA$ , by I. 32. Therefore the angle  $EBC$  is equal to twice the angle  $BAC$ . Therefore the angle  $EBC$  is equal to the angle  $BEC$ ; therefore  $EC$  is equal to  $BC$ , by I. 6. Thus  $EBC$  is equilateral.

Again,  $EC$  is equal to  $ED$ ; therefore the angle  $BEC$  is equal to twice the angle  $BDC$ . Therefore the angle  $BDC$  is equal to the angle  $BAC$ . Therefore the angle  $ACD$  is equal to the angle  $ABC$ , by I. 32. Then the triangle  $ABC$  is equiangular to the triangle  $ACD$ .

475. Take the diagram of I. 43. Let  $BD$  and  $HG$  intersect at  $L$ . Through  $L$  draw a straight line parallel to  $AD$ , meeting  $AB$  at  $M$  and  $DC$  at  $N$ .

The complements  $AL$  and  $LC$  are equal, by I. 43; to each add  $KN$ . Then the parallelograms  $AK$  and  $MF$  together are equal to the parallelogram  $KC$ . Therefore the difference of  $KC$  and  $AK$  is equal to  $MF$ .

Now the parallelogram  $MK$  is twice the triangle  $BLK$ , and the parallelogram  $LF$  is twice the triangle  $DLK$ , by I. 41; therefore the parallelogram  $MF$  is twice the triangle  $BKD$ . Therefore the difference of the parallelograms  $KC$  and  $AK$  is equal to twice the triangle  $BKD$ .

476. Take  $BC$  equal to the given side; draw  $CD$  at right angles to  $BC$ , and make it equal to the difference between the hypotenuse and the other side. At the point  $B$  in the straight line  $BD$  make the angle  $DBE$  equal to the angle  $BDC$ , and let  $BE$  meet  $DC$  produced at  $A$ : then  $ABC$  is the triangle required.

For the angle  $ABD$  is equal to the angle  $ADB$ , by construction; therefore  $AD$  is equal to  $AB$ , by I. 6; but  $AD$  exceeds  $AC$  by  $CD$ ; therefore  $AB$  exceeds  $AC$  by  $CD$ . Thus the hypotenuse  $AB$  exceeds the side  $AC$  by the prescribed length. Also the side  $CB$  has the prescribed length.

The process requires the angle  $CBD$  to be less than the angle  $CDB$ ; it will be found that this leads to the condition that the hypotenuse must be less than the sum of the sides, which is of course necessary by I. 20.

477. The triangle  $EBC$  is half the triangle  $ABC$ , by I. 38; the triangle  $BED$  is half the triangle  $EBC$ , and is therefore one-fourth of the triangle  $ABC$ . Bisect  $AG$  at  $H$ ; then the triangle  $ABH$  is equal to the triangle  $GBH$ , and the triangle  $AHE$  is equal to the triangle  $GHE$ : therefore the

triangle  $BHE$  is half the triangle  $BAE$ , and is therefore equal to the triangle  $BDE$ . Therefore  $GH$  is equal to  $GD$ , by Exercise 114. Thus  $AG$  is double of  $GD$ .

478. Let  $BC$  meet  $AE$  at  $F$ . The angle  $DAC$  is equal to the angle  $DCA$ ; see Exercise 59. Suppose  $F$  to be between  $D$  and  $C$ . The angle  $EFC$  is equal to the angles  $EDF$  and  $DEF$ , by I. 32. Also the angle  $EFC$  is equal to the angles  $DCA$  and  $FAC$ , by I. 32; that is to the angles  $DAC$  and  $FAC$ ; that is to twice the angle  $FAC$  together with the angle  $DAF$ ; that is to the angles  $BAC$  and  $DAF$ . Thus the sum of the angles  $EDF$  and  $DEF$  is equal to the sum of the angles  $BAC$  and  $DAF$ . But the right angles  $EDF$  and  $BAC$  are equal; therefore the angle  $DAF$  is equal to the angle  $DEF$ ; therefore  $DA$  is equal to  $DE$ , by I. 6.

479. The triangle  $ACF$  is half the rhombus, and the triangle  $ABC$  is half the square; therefore these triangles are equal; therefore  $FEB$  is a straight line parallel to  $AC$ , by I. 39. Draw  $BH$  and  $EK$  perpendicular to  $AC$ . Then  $BH$  is equal to  $AH$  by I. 6; and therefore equal to half  $AC$ ; and therefore equal to half  $AE$ . Therefore  $EK$  is half of  $AE$ , by I. 34. And the angle  $AKE$  is a right angle. Therefore if  $EK$  be produced to a point  $L$  such that  $KL$  is equal to  $KE$ , the triangle  $ALE$  is equilateral. Thus  $AEK$  is an angle of an equilateral triangle, and therefore it is two-thirds of a right angle.

Therefore the angle  $KAE$  is one-third of a right angle, by I. 32. Also the angle  $BAC$  is half a right angle; therefore the angle  $BAE$  is one-sixth of a right angle. The angles  $CAF$  and  $EAF$  are equal by Exercise 11; therefore each of them is one-sixth of a right angle. Thus the angles  $BAE$ ,  $EAF$ , and  $FAC$  are all equal.

480. From  $G$  draw  $GM$  perpendicular to  $AB$ , and  $GN$  perpendicular to  $AC$ . Then the angle  $NGM$  is a right angle. The angle  $EGD$  is also a right angle. Therefore the angle  $EGN$  is equal to the angle  $DGM$ .

In the two triangles  $EGN$  and  $DGM$  the side  $EG$  is equal to the side  $DG$ ; the angles  $EGN$  and  $DGM$  have been shewn to be equal; and the right angles  $ENG$  and  $DMG$  are equal: therefore  $GM$  is equal to  $GN$  by I. 26. Thus  $AMGN$  is a square, and the diagonal  $AG$  bisects the angle  $BAC$ ; so that the locus of  $G$  is the straight line which bisects the angle  $BAC$ .

481. We shall first shew that a rectangle is greater than any parallelogram on the same base with the same perimeter.

Let  $ABCD$  be a parallelogram, and  $ABEF$  a rectangle on the same base  $AB$ , and having the same perimeter; then  $AF$  is equal to  $AD$  and therefore the perpendicular from  $D$  on  $AB$  is less than  $AF$ . Let  $DC$  produced cut  $AF$  at  $G$  and  $BE$  at  $H$ . Then the parallelogram  $ABCD$  is equal to  $ABHG$ , by I. 35; and is therefore less than the rectangle  $ABEF$ .

Next we shall shew that a square is greater than any rectangle having the same perimeter.

Let  $ABCD$  be a rectangle,  $AB$  being longer than  $AD$ . Let  $AEFG$  be a square having the same perimeter,  $E$  being on  $AB$ , and  $G$  on  $AD$  produced.

Since the perimeters are equal  $EB$  is equal to  $GD$ , and  $BC$  is less than  $GF$ ; therefore the rectangle  $EC$  is less than the rectangle  $DF$ ; therefore the rectangle  $AC$  is less than the square  $EG$ .

482. Let  $ABCD$  be the square in which the square of given area is to be inscribed. Join  $AC$ ; and bisect it at  $E$ . From  $E$  as centre with a radius equal to half the diagonal of the square of given area describe a circle cutting  $AB$  at  $F$ ; and produce  $FE$  to meet  $CD$  at  $G$ . Through  $E$  draw a straight line at right angles to  $FG$ , meeting  $BC$  at  $H$ , and  $AD$  at  $K$ . Then  $FHGK$  will be the square required.

For  $FE$  is equal to  $GE$  and  $HE$  is equal to  $KE$ , by Exercise 36. Then from the triangles  $FEH$  and  $GEH$  we shew that  $FH$  is equal to  $GH$ ; and from the triangles  $HEG$  and  $KEG$  we shew that  $HG$  is equal to  $KG$ . In this way we shew that the figure  $FHGK$  is equilateral.

The angle  $AEB$  is a right angle, and so also is the angle  $FEH$ ; therefore the angle  $AEF$  is equal to the angle  $BEH$ . Also the angle  $EAF$  is equal to the angle  $EBH$ , each being half a right angle. And  $EB$  is equal to  $EA$ . Therefore  $EF$  is equal to  $EH$ ; therefore the angle  $EFH$  is equal to the angle  $EHF$ ; therefore each of them is half a right angle. In this way we shew that the figure  $FHGK$  is rectangular.

483. The triangle  $BDC$  is twice the triangle  $ADC$ , and the triangle  $BDF$  is twice the triangle  $ADF$ ; see I. 38. Therefore the triangle  $BFC$  is twice the triangle  $FAC$ . Similarly the triangle  $BFC$  is twice the triangle  $FAB$ . Therefore the triangle  $BFC$  is equal to the sum of the triangles  $FAC$  and  $FAB$ ; so that the triangle  $BFC$  is half the triangle  $BAC$ .

Again the triangle  $DAC$  is equal to the triangle  $EAB$ , each being one-third of the triangle  $ABC$ . Take away from each the figure  $AEDF$ ; thus the triangle  $FEC$  is equal to the triangle  $FDB$ . But the triangle  $FAC$  was shewn to be equal to the triangle  $FAB$ ; therefore the triangle  $AEF$  is equal to the triangle  $ADF$ . Therefore the figure  $ADEF$  is twice the triangle  $ADF$ . But the triangle  $BDF$  is twice the triangle  $ADF$ . Therefore the figure  $ADEF$  is equal to the triangle  $BDF$ .

484. On  $AB$  take  $AF$  equal to  $AE$ , and on  $BF$  take  $BG$  equal to  $BD$ . Then the triangle  $AOF$  is equal to the triangle  $AOE$ , and the triangle  $BOG$  is equal to the triangle  $BOD$ , by I. 4.

We shall now shew that the triangle  $FOG$  is equal in area to the triangle  $DOE$ .

The angle  $EOA$  is equal to the angles  $OAB$  and  $OBA$ , by I. 32; that is to half the sum of the angles  $BAC$  and  $ABC$ , that is to half a right angle. Therefore the angle  $AOE$  is half a right angle. Similarly the angle  $BOD$  is half a right angle, and also the angle  $BOG$ . Hence the angle  $GOF$  is half a right angle, and the angle  $DOE$  is three halves of a right angle. On  $OA$  take  $OH$  equal to  $OD$ , and join  $EH$ . Then in the triangles  $EOH$  and  $FOG$  the side  $EO$  is equal to the side  $FO$ ; the side  $HO$  is equal to the side  $GO$ ; and the angle  $HOE$  is equal to the angle  $GOF$  each being half a right angle. Therefore the triangles  $EOH$  and  $FOG$  are equal in all respects. But the triangle  $EOH$  is equal in area to the triangle  $EOD$ , by I. 38; therefore the triangle  $FOG$  is equal in area to the triangle  $EOD$ .

Thus the triangle  $AOF$  is equal to the triangle  $AOE$ , the triangle  $BOG$  is equal to the triangle  $BOD$ , and the triangle  $GOF$  is equal to the triangle  $DOE$ ; therefore the triangle  $AOB$  is half the quadrilateral  $ABDE$ .

485. Let  $ABC$  be the scalene triangle. If possible let  $BD$  be the dividing straight line. The angle  $BCD$  is not equal to the angle  $BAD$ , since by supposition the triangle is scalene; the angle  $BCD$  cannot be

equal to the angle  $BDA$ , by I. 32; and thus the only possible case is that the angle  $BCD$  should be equal to the angle  $ABD$ , and the angle  $BAD$  equal to the angle  $CBD$ . This requires  $AB$  to be equal to  $BC$ , and is contrary to the supposition that the triangle is scalene.

486. Let  $AE$  and  $CD$  intersect at  $G$ . Then  $CG$  is equal to  $GD$ , by Exercise 78. Therefore the triangle  $CGF$  is equal to the triangle  $DGF$ , and the triangle  $CGE$  is equal to the triangle  $DGE$ , by I. 38. Therefore the triangle  $FCE$  is equal to the triangle  $DFE$ . But  $BC$  is equal to  $CE$ , by supposition; therefore the triangle  $BCF$  is equal to the triangle  $ECF$ , by I. 38. Therefore the triangle  $BFE$  is double the triangle  $DFE$ . Therefore  $BF$  is twice  $DF$ .

487.  $BE$  and  $AD$  are each equal and parallel to  $CZ$ ; therefore  $BE$  and  $AD$  are equal and parallel; therefore  $AB$  and  $ED$  are equal and parallel, by I. 33, and  $ADEB$  is a parallelogram.

Produce  $ZC$  to meet  $AB$  at  $L$ . Then the parallelogram  $DL$  is equal to the parallelogram  $AZ$  by I. 35, which is equal to the parallelogram  $FC$ . Similarly  $EL$  is equal to  $KC$ ; thus  $ADEB$  is equal to the sum of  $FC$  and  $CK$ .

488. In the quadrilateral  $ABCD$  suppose that  $AB$  is parallel to  $DC$ . Let  $AC$  and  $BD$  intersect at  $E$ . Through  $E$  draw a straight line parallel to  $AB$  meeting  $AD$  at  $M$ , and  $BC$  at  $N$ . Then  $MN$  will be bisected at  $E$ . For if  $ME$  be not equal to  $NE$  one of them must be the greater; suppose  $ME$  greater than  $NE$ . Then the triangle  $MAE$  will be greater than the triangle  $NBE$ , and the triangle  $MDE$  will be greater than the triangle  $NCE$ : see I. 38. Therefore the triangle  $AED$  is greater than the triangle  $BEC$ .

Again, the triangle  $ABC$  is equal to the triangle  $ABD$ , by I. 37; therefore the triangle  $AED$  is equal to the triangle  $BEC$ . But the triangle  $AED$  was shewn to be greater than the triangle  $BEC$ . Therefore  $ME$  and  $NE$  cannot be unequal; that is, they are equal.

489. Let  $ABC$  and  $DEF$  be two triangles; let the bases  $AB$  and  $DF$  be equal, and in the same straight line, and let  $CE$  be parallel to this straight line. Let a straight line be drawn parallel to  $CE$ , meeting  $CA$  at  $G$ ,  $CB$  at  $H$ ,  $ED$  at  $K$ , and  $EF$  at  $L$ . Then  $GH$  will be equal to  $KL$ .

For if  $GH$  be not equal to  $KL$  one of them must be the greater; suppose  $GH$  the greater, and from it cut off  $GM$  equal to  $KL$ . Join  $AM$ ,  $BM$ ,  $CM$ .

Then the triangle  $GCM$  is equal to the triangle  $KEL$ , the triangle  $GMA$  is equal to the triangle  $KLD$ , and the triangle  $AMB$  is equal to the triangle  $DLF$ , by I. 38. Therefore the triangle  $DEF$  is equal to the sum of the triangles  $GCM$ ,  $GMA$ , and  $AMB$ ; but the triangle  $ACB$  is equal to the triangle  $DEF$ ; therefore the triangle  $ACB$  is equal to the sum of the triangles  $GCM$ ,  $GMA$ , and  $AMB$ ; that is the whole is equal to a part which is absurd. Therefore  $GH$  is not unequal to  $KL$ , that is  $GH$  is equal to  $KL$ .

490. Let  $O$  be the middle point of  $BC$ ; then  $OA$ ,  $OB$ , and  $OC$  are all equal, by Exercise 59. Now  $AB$  is equal to half  $AC$ , and is therefore less than  $OB$ , which is half  $BC$ ; therefore the angle  $OBA$  is greater than the angle  $BOA$ , by I. 18. But the angle  $BOA$  is equal to the angles  $OCA$  and  $OAC$ , by I. 32; therefore the angle  $BOA$  is equal to twice the angle  $OCA$ , by I. 5. Therefore the angle  $OBA$  is greater than twice the angle  $OCA$ .

491. Let  $ABCD$  be the parallelogram. Trisect  $DC$  at  $E$  and  $F$ ; and trisect  $BC$  at  $H$  and  $G$ : see Exercise 70. Then the triangles  $ADC$  and  $ABC$  are equal, by I. 34. Thus each of the triangles  $DAE$ ,  $EAF$ ,  $FAC$ ,  $CAG$ ,  $GAH$ ,  $HAB$  is one-sixth of the parallelogram  $ABCD$ . Thus the triangles  $DAF$ , the quadrilateral  $FAGC$ , and the triangle  $GAB$  are all equal, each being one-third of the parallelogram  $ABCD$ .

492. The angles  $DAB$  and  $ABH$  are together equal to two right angles, by I. 29; therefore the angles  $DAB$  and  $AHB$  are together equal to two right angles, by I. 5. But the angles  $AHC$  and  $AHB$  are together equal to two right angles, by I. 13; therefore the angle  $DAB$  is equal to the angle  $AHC$ . Similarly the angle  $DAB$  is equal to the angle  $AKC$ . Also the angle  $DAB$  is equal to the angle  $HCK$ , by I. 34. Therefore three times the angle  $BAD$  is equal to the sum of the angles  $AHC$ ,  $AKC$ ,  $HCK$ ; that is to the sum of the angles of the triangle  $HKC$  together with the angles  $AHK$  and  $AKH$ ; that is to two right angles together with two angles of an equilateral triangle; that is to ten-thirds of a right angle. Therefore the angle  $DAB$  is ten-ninths of a right angle.

493. Let  $ABC$  be the given triangle;  $D$  a given point in the side  $AC$ . Draw a straight line  $DE$  meeting  $AC$  at  $D$  and  $AB$  at  $E$ , so that the triangle  $DAE$  may be one-third of the given triangle: this can be done by aid of I. 44,  $DA$  being the given straight line,  $A$  the given angle, and  $DE$  the diagonal of the parallelogram so described, which will be two-thirds of the triangle  $ABC$ . Then bisect the quadrilateral  $DEBC$  by a straight line drawn from  $D$ , by Exercise 123. Thus the triangle  $ABC$  is divided into three equal parts by straight lines drawn from  $D$ .

If the triangle  $DAB$  be less than a third of the triangle  $ABC$ , then  $DC$  and the angle  $C$  must be used in the first part of the construction instead of  $DA$  and the angle  $A$ .

494. Let  $AC$  and  $BF$  intersect at  $H$ , let  $BE$  and  $CD$  intersect at  $G$ . Let  $GH$ , produced if necessary, meet  $BC$  at  $K$ . Then will  $BK$  be equal to  $CK$ .

Through  $H$  draw a straight line parallel to  $BC$ , meeting  $BE$  at  $M$  and  $CD$  at  $N$ . The triangles  $ACD$  and  $EBF$  are equal, by I. 38. Therefore  $HN$  is equal to  $HM$ , by Exercise 489. Therefore the triangle  $GHN$  is equal to the triangle  $GHM$ , by I. 38. Therefore  $BK$  is equal to  $CK$ , by Exercise 489.

## II. 1 to 14.

495. Let  $ABC$  be the given triangle,  $BC$  the side to be produced. Suppose that  $CA$  is not less than  $BA$ . From  $A$  draw  $AD$  perpendicular to  $BC$  or  $BC$  produced. Produce  $CD$  to  $E$  so that  $DE$  may be equal to  $DC$ . Then the difference of the squares on  $CA$  and  $BA$  is equal to the difference of the squares on  $CD$  and  $BD$ , by I. 47; that is to the rectangle of the sum and difference of  $CD$  and  $BD$ , by page 269 of the *Euclid*. Now if the perpendicular falls *within* the triangle  $ABC$  the sum of  $CD$  and  $BD$  is  $BC$ , and the difference is  $BE$ ; and if the perpendicular falls *without* the triangle  $ABC$  the sum of  $CD$  and  $BD$  is  $BE$ , and the difference is  $BC$ . Thus in each case the difference of the squares on  $CD$  and  $BD$  is equal to the rectangle  $BC, BE$ ; so that  $BE$  is the produced part required.

496. Let  $AB$  be the given straight line; let  $BC$  represent the produced part. Then we require that the squares on  $AB$  and  $BC$  shall be equal to twice the rectangle  $AC, BC$ ; so that the square on  $BC$  together with twice the square on  $AB$  must be equal to the square on  $AB$  together with twice the rectangle  $AC, BC$ ; that is the square on  $BC$  together with twice the square on  $AB$  must be equal to the squares on  $AC$  and  $BC$ , by II. 7; therefore the square on  $AC$  must be equal to twice the square on  $AB$ . This determines  $AC$ , and shews that  $AC$  must be equal to the diagonal of a square described on the side  $AB$ .

497. Let  $AB$  be the given straight line; let  $BC$  represent the produced part. Then we require that the squares on  $AB$  and  $AC$  shall be equal to twice the rectangle  $AC, BC$ ; so that the square on  $AC$  together with twice the square on  $AB$  must be equal to the square on  $AB$  together with twice the rectangle  $AC, BC$ ; that is the square on  $AC$  together with twice the square on  $AB$  must be equal to the squares on  $AC$  and  $BC$ , by II. 7; therefore the square on  $BC$  must be equal to twice the square on  $AB$ . This determines  $BC$ , and shews that  $BC$  must be equal to the diagonal of a square described on the side  $AB$ .

498. In the diagram and proof of II. 11 it is shewn that the rectangle  $CF, FA$  is equal to the square on  $CA$ . If then  $CA$  be the given straight line describe a square on  $CA$ , and proceed as in II. 11; then  $F$  will be the point required.

499. Take any straight line  $AD$ ; produce  $AD$  to  $B$  so that  $DB$  may be equal to twice  $AD$ . From  $A$  draw a straight line  $AE$  at right angles to  $AB$ . From the centre  $D$  with radius equal to  $DB$  describe a circle cutting  $AE$  at  $C$ . Then  $DBC$  will be such a triangle as is required.

For the square on  $BC$  is equal to the squares on  $CD, DB$  and twice the rectangle  $AD, DB$ . But twice the rectangle  $AD, DB$  is equal to the square on  $DB$ , because  $DB$  is equal to twice  $AD$ . Also  $CD$  is equal to  $DB$ . Therefore the square on  $BC$  is equal to three times the square on  $DB$ .

500. Let  $ABC$  be a triangle, and  $AD$  a perpendicular to the base  $BC$  produced; and suppose that the square on  $AB$  exceeds the squares on  $BC, CA$  by the rectangle  $BC, CA$ .

The square on  $AB$  exceeds the squares on  $BC, CA$  by twice the rectangle  $BC, CD$ ; therefore the rectangle  $BC, CA$  is equal to twice the rectangle  $BC, CD$ ; therefore  $CA$  is equal to twice  $CD$ . Produce  $CD$  to  $E$ , so that  $DE$  is equal to  $CD$ ; and join  $AE$ . Then  $AC$  is equal to  $CE$ . Also  $AE$  is equal to  $AC$ , by I. 4. Therefore  $ACE$  is an equilateral triangle. Therefore the angle  $ACE$  is one-third of two right angles, by I. 32. Therefore the angle  $ACB$  is two-thirds of two right angles, by I. 13.

501. Let the straight line  $AB$  be the sum of two adjacent sides of the rectangle; let  $CD$  represent the difference of the two sides. Then  $AB$  is known, and we proceed to find  $CD$ .

The difference of the squares on  $AB$  and  $CD$  is equal to the rectangle contained by the sum and the difference of  $AB$  and  $CD$ ; that is to four times the rectangle which is required; that is to four times the given square; that is to a known quantity. But the square on  $AB$  is known; thus the square on  $CD$  is known; and therefore  $CD$  is known. Hence the sides

of the required rectangle are found; for one side is half the sum of  $AB$  and  $CD$ , and the other is half the difference of  $AB$  and  $CD$ .

502. Use the same notation as in Exercise 501. Then the difference of the squares on  $AB$  and  $CD$  is equal to four times the given square. But  $CD$  is known; hence the square on  $AB$  is known, and therefore  $AB$  is known. Hence the sides of the required rectangle are found.

503. Let  $ABCD$  be the given square; let  $H, E, F, G$  be the middle points of the successive sides, so that  $HEFG$  is a square. Let  $PQRS$  be any other inscribed square. In each square the diagonals intersect at the same point, say  $T$ : see Exercise 36.

Then the square on  $PQ$  is equal to the squares on  $TP$  and  $TQ$ , and the square on  $HE$  is equal to the squares on  $TH$  and  $TE$ , by I. 47. But  $TP$  is greater than  $TH$ , and  $TQ$  is greater than  $TE$ , by I. 19; therefore the square on  $PQ$  is greater than the square on  $HE$ .

504. Let  $AB$  be the given straight line; it is required to divide it at  $C$ , so that the squares on  $AB$  and  $BC$  may be equal to twice the square on  $AC$ .

We require then that three times the square on  $AB$ , together with the square on  $BC$ , may be equal to twice the square on  $AB$  together with twice the square on  $AC$ . Produce  $BA$  to  $D$ , so that  $AD$  may be equal to  $AC$ . Then  $DB$  is the sum of  $AB$  and  $AC$ , and  $CB$  is the difference; so that the square on  $DB$  together with the square on  $CB$  is equal to twice the square on  $AB$  together with twice the square on  $AC$ ; by page 269 of the *Euclid*. Hence we must have the square on  $DB$  equal to three times the square on  $AB$ . Thus  $DB$  is known; then we take  $AC$  equal to  $AD$ ; and the straight line  $AB$  is divided at  $C$  in the manner required.

505. Let  $ABDC$  and  $AGFE$  be rectangles of equal areas and perimeters. Place them so as to have a common angle at  $A$ ; and let  $AE$  fall on  $AC$ , and  $AB$  on  $AG$ . Let  $BD$  and  $EF$  intersect at  $H$ .

Then since the perimeters are equal we have twice  $EC$  equal to twice  $HF$ , and therefore  $EC$  is equal to  $HF$ . Also since the areas are equal the rectangle  $BGFH$  is equal to the rectangle  $ECDH$ . But  $EC$  is equal to  $HF$ . Therefore  $CD$  is equal to  $FG$ . Therefore by using either the condition that the areas are equal or that the perimeters are equal we find that  $AC$  is equal to  $AG$ . Thus the two rectangles are equal in all respects.

506. Let  $O$  be the centre of the rectangle. Since the sum of  $PA$  and  $PC$  is equal to the sum of  $PB$  and  $PD$ , the squares on  $PA$  and  $PC$  together with twice the rectangle  $PA, PC$  are equal to the squares on  $PB$  and  $PD$  together with twice the rectangle  $PB, PD$ , by II. 4. But the squares on  $PA$  and  $PC$  are equal to twice the square on  $PO$  and twice the square on  $AO$ ; and the squares on  $PB$  and  $PD$  are equal to twice the square on  $PO$  and twice the square on  $BO$ , see the *Euclid*, p. 293: so that the squares on  $PA$  and  $PC$  are equal to the squares on  $PB$  and  $PD$ . Hence the rectangle  $PA, PC$  is equal to the rectangle  $PB, PD$ .

Thus these two rectangles have equal areas and equal perimeters; and therefore, by Exercise 505, must be equal in all respects. Hence  $PA$  must be equal either to  $PB$  or  $PD$ . If we take  $PA$  equal to  $PB$  the point  $P$  falls on the straight line which is parallel to  $BC$  and passes through the centre of the rectangle. If we take  $PA$  equal to  $PD$  the point  $P$  falls on the

straight line which is parallel to  $AB$  and passes through the centre of the rectangle.

### III. 1 to 37.

507. Let  $BC$  be the straight line which is to be touched at the point  $A$ . Let  $D$  be the other given point. From  $A$  draw  $AE$  at right angles to  $BC$ . At the point  $D$  make the angle  $ADF$  equal to the angle  $DAE$ , and let  $DF$  intersect  $AE$  at  $G$ .

Then  $GA$  is equal to  $GD$  by I. 6. Therefore a circle described from the centre  $G$  with radius  $GA$  will pass through  $D$ ; and it will touch the straight line  $BC$  by III. 16, Corollary.

508. Let  $C$  be the centre of the given circle,  $A$  the point at which it is to be touched; let  $B$  be the other given point through which the required circle is to pass.

Join  $AB$  and bisect it at right angles by the straight line  $EF$ ; the centre of the required circle must be on the straight line  $EF$ , by III. 1. Also the centre of the circle must be on the straight line  $CA$ , or  $CA$  produced, by III. 11 and III. 12. Therefore the centre of the circle must be at the intersection of  $CA$  and  $EF$ . Thus the circle is determined.

509. Let  $A$  be the given point,  $AB$  a tangent to the given circle at that point,  $CD$  the given straight line. The centre of the required circle must lie on the straight line  $AE$  drawn through  $A$  at right angles to  $AB$ . As the circle is to touch  $AB$  and  $CD$  its centre must lie on the straight line which bisects one of the two angles made by  $AB$  and  $CD$ . See the note on III. 17 in the *Euclid*, page 275. Thus by the intersection of the straight line  $AE$  with the straight lines bisecting these two angles the position of the centre is found and the circle determined. We see that there are two solutions.

510. A circle will go round  $ABDE$ , by page 276 of the *Euclid*; therefore the angle  $ABE$  is equal to the angle  $ADE$ , by III. 21. Now the angle  $FBD$  is the difference between a right angle and the angle  $BDF$ , by I. 32; that is the difference between a right angle and the angle  $EDC$ , by I. 15; so that the angle  $FBD$  is equal to the angle  $ADE$ , and therefore to the angle  $ABE$ .

511. A circle described on  $BC$  as diameter will go through  $E$  and  $F$ , by page 276 of the *Euclid*. Then  $EK$  and  $FK$  will each be equal to half  $BC$ , by Exercise 59; so that  $K$  is the centre of this circle.

The angle  $FKE$  is equal to twice the angle  $FCE$  by III. 20. Suppose a perpendicular drawn from  $K$  on  $FE$ ; thus the triangle  $FKE$  is divided into two triangles equal in all respects; and the angle  $FEK$  is seen to be equal to the difference of a right angle and half the angle  $FKE$ ; that is to the difference of a right angle and the angle  $FCE$ . But the angle  $FAC$  is also equal to the difference of a right angle and the angle  $FCE$ . Therefore the angle  $FAC$  is equal to the angle  $FEK$ ; and therefore also to the angle  $EFK$ .

512. Let  $AC$ ,  $AD$  be on the same side of  $AB$ , and  $AC$  the nearer to  $AB$ . The angle  $CDF$  is equal to the angle  $CBA$ , by I. 13 and III. 22. The angle  $CEF$  is equal to the angles  $CAB$  and  $ABE$ , by I. 32. Therefore the sum of the angles  $CDF$  and  $CEF$  is equal to the sum of the angles  $CBA$ ,  $CAB$  and  $ABE$ . But the angles  $CBA$  and  $CAB$  are together equal to a right angle, by I. 32 and III. 31; and the angle  $ABE$  is a right angle by III. 16. Therefore the sum of the angles  $CDF$  and  $CEF$  is equal to two right angles.



Therefore a circle will go round  $FDCE$ , by page 276 of the *Euclid*; therefore the angle  $FDE$  is equal to the angle  $FCE$ , by III. 21.

The proof is very similar for other forms of the diagram.

513. Let  $ABCD$  be a quadrilateral. Bisect the angles  $A, B, C, D$  by straight lines which meet and form a quadrilateral: then a circle can be described round this quadrilateral.

Let  $AP$  and  $BP$  be two of the straight lines, and  $CQ$  and  $DQ$  the other two straight lines. Then the angle  $APB$ , half the angle  $A$ , and half the angle  $B$  are together equal to two right angles, by I. 32. So also the angle  $CQD$ , half the angle  $C$ , and half the angle  $D$  are together equal to two right angles. Therefore the angles  $APB$  and  $DQC$  together with half the sum of the angles  $A, B, C, D$  are together equal to four right angles. But half the sum of the angles  $A, B, C, D$  is equal to two right angles. Hence the angles  $APB$  and  $DQC$  are together equal to two right angles. Hence a circle can be described round the quadrilateral of which  $P$  and  $Q$  are opposite corners, by page 276 of the *Euclid*.

514. Let  $A$  be the centre of one circle,  $B$  the centre of the other; join  $AB$  meeting the first circle at  $G$  and the second at  $H$ . Then  $GH$  is the shortest distance between the circles.

For draw the straight line  $CD$  from any point  $C$  on the first circle to any point  $D$  on the second. Join  $AC, AD$ , and  $DB$ . Then  $AC$  and  $CD$  are together greater than  $AD$ , by I. 20; therefore  $AC, CD$ , and  $DB$  are together greater than  $AD$  and  $DB$  together. But  $AD$  and  $DB$  together are greater than  $AB$ ; therefore  $AC, CD$  and  $DB$  together are greater than  $AB$ . But  $AC$  is equal to  $AG$ , and  $BD$  is equal to  $BH$ ; therefore  $CD$  is greater than  $GH$ .

515. Suppose the straight line drawn; let  $BAC$  denote it. Join  $B$  and  $C$  with  $D$  the other extremity of the common chord. Then the angles  $ABD$  and  $ACD$ , being the angles in known segments of circles are known; also the side  $BC$  is known. Hence the triangle  $BCD$  can be constructed. Then the angle  $BAD$  becomes known, being the angle subtended by a known chord  $BD$  in a known circle. This determines the situation of the straight line  $BAC$ .

There will be in general two solutions, in one of which the segment  $BAD$  will be greater than a semicircle, and in the other, less.

516. Suppose the polygon to be a quadrilateral. Then the sum of the alternate angles is equal to two right angles, by III. 22; and the sum of the alternate angles, together with two right angles, is equal to four right angles.

Suppose the polygon to be a hexagon, as  $ABCDEF$ . Draw  $AD$ . The sum of the angles  $ABC$  and  $ADC$  is two right angles, by III. 22; so also is the sum of the angles  $ADE$  and  $AFE$ . Therefore the sum of the angles  $ABC, CDE, EFA$  is equal to four right angles; and the sum of these angles, together with two right angles, is equal to six right angles.

Again, suppose the polygon to be an octagon, as  $ABCDEFGH$ . Draw the straight lines  $AD$  and  $AF$ . Then the angles  $ABC$  and  $ADC$  are equal to two right angles, so are the angles  $ADE$  and  $AFE$ , so are the angles  $AFG$  and  $AHG$ . Then as before we obtain the required result.

In this way the proposition may be established for a polygon of any even number of sides.

517. Let  $C$  be the centre of the given circle,  $A$  the given point on the circumference,  $HK$  the given chord. On  $AC$  as diameter describe a semicircle, and let it cut  $HK$  at  $B$ . Join  $AB$  and produce it to meet the given circle at  $D$ .

The angle  $ABC$  is a right angle, by III. 31; therefore  $AD$  is bisected at  $B$ , by III. 3.

There will be two solutions in general, as the semicircle may cut  $HK$  at two points; but no solution if the semicircle does not meet  $HK$ .

518. Let  $A, B, C, D, E, \dots$  be the successive angular points of the polygon;  $O$  the centre of the circle.

The angles at  $A, B, C, D, \dots$  are bisected respectively by the straight lines  $OA, OB, OC, OD, \dots$ : see the note on III. 17 on page 275 of the *Euclid*. Then in the triangles  $OBA, OBC$  the side  $OB$  is common, the angle  $OBA$  equal to the angle  $OBC$ , and the side  $BA$  equal to the side  $BC$ , by supposition: therefore the angle  $BAO$  is equal to the angle  $BCO$  by I. 4. Therefore the doubles of these are equal; so that the whole angle at  $A$  is equal to the whole angle at  $C$ .

In this way we shew that the alternate angles are equal; and so if the number of them is *odd* they are all equal. If the number of the angles is even they are not necessarily equal; for instance, a circle might be inscribed in any rhombus.

519. If  $AE$  and  $BD$  intersect *within* the circle the angle  $APB$  is measured by half the *sum* of the arc  $DE$  and a semicircumference, by page 294 of the *Euclid*, and is therefore constant. Similarly if  $AE$  and  $BD$  intersect *without* the circle the angle  $APB$  is measured by half the *difference* of the arc  $DE$  and a semicircumference and is therefore constant.

520. Let  $ABC$  be one of the triangles. The four angles of the quadrilateral  $AD$  are together equal to four right angles; hence the angle  $BDC$  is equal to the excess of two right angles above the angle  $BAC$  and is therefore a constant angle. Hence the locus of  $D$  is a segment of a circle by page 276 of the *Euclid*. The straight lines which bisect the angle  $BDC$  all meet at a point, by Exercise 230.

521. The angle  $OBA$  is equal to the angle  $OAB$ , by I. 5; that is the angle  $OBC$  is a constant angle, by III. 21. Hence the locus of  $B$  is a circle, by page 276 of the *Euclid*. This circle is made up of two segments each equal to the corresponding segment of the given circle cut off by  $OC$ : see III. 24.

522. Suppose  $PE$  drawn perpendicular to  $AB$ , and produce  $EP$  to meet  $DC$ ; then these straight lines will meet at right angles by I. 29. Thus  $PE$  and  $PG$  form one straight line.

Since the angles  $PFB$  and  $PEB$  are right angles a circle will go round  $PFBE$ , by page 276 of the *Euclid*; therefore the angle  $PEF$  is equal to the angle  $PBF$ . Similarly the angle  $PGH$  is equal to the angle  $PDH$ . But  $BC$  is parallel to  $AD$ ; and therefore the angle  $PBF$  is equal to the angle  $PDH$ , by I. 29.

Therefore the angle  $PEF$  is equal to the angle  $PGH$ . Therefore  $EF$  is parallel to  $GH$ , by I. 27.

523. Let  $AB$  be a fixed chord of a circle,  $C$  a fixed point in it. Let  $PCQ$  be any other chord of the circle. Let  $D$  be the middle point of  $AB$  and  $R$  the middle point of  $PQ$ . Let  $O$  be the centre of the circle,

The angles  $ODC$  and  $ORC$  are right angles, by III. 3; therefore a circle will go round  $ORCD$ , by page 276 of the *Euclid*; therefore the angle  $CRD$  is equal to the angle  $COD$ , by III. 21. Thus the angle  $CRD$  is a constant angle.

524. The angle  $BFA$  is equal to the angle  $BEC$ , because the segments are similar. For the same reason the angle  $ADB$  is equal to the angle  $CDB$ ; therefore the angle  $BDE$  is equal to the angle  $BDF$ , by I. 13. Therefore the angle  $BCE$  is equal to the angle  $BAF$ , by III. 22. Thus the angle  $BFA$  is equal to the angle  $BEC$ , and the angle  $BAF$  is equal to the angle  $BCE$ ; therefore the angle  $ABF$  is equal to the angle  $CBE$ , by I. 32. Thus the triangle  $BEC$  is equiangular to the triangle  $BFA$ .

Also the angle  $BFA$  is equal to the angle  $BDA$ , by III. 21; that is to the angle  $BCE$ , by I. 13 and III. 22; that is to the angle  $BAF$ , as shewn above. Thus the triangle  $BAF$  is isosceles by I. 6. Therefore also the triangle  $BCE$  is isosceles.

525. Let  $A$  and  $B$  be the centres of the circles,  $AC$ ,  $BD$  perpendiculars on the common tangent. From  $O$  the middle point of  $AB$  draw  $OQ$  perpendicular to  $CD$ . Then  $OQ$  is equal to half the sum of  $CA$  and  $BD$ , by Exercise 89; that is  $OQ$  is equal to half  $AB$ , and is therefore constant. Hence  $CD$  touches at  $Q$  the circle described from  $O$  as centre with radius equal to half  $AB$ .

526. Suppose  $A$  the given point, and  $BC$  the given straight line. Suppose that  $P$  and  $Q$  are two points in  $BC$ , such that  $PQ$  is of the given length and the angle  $PAQ$  equal to the given angle. Suppose that a circle goes round  $PAQ$ , and that  $O$  is the centre of the circle. The radius of this circle is known, for it is the circle in which a chord of given length subtends a given angle at the circumference. Since the radius and the length of  $PQ$  are known the distance of  $O$  from the fixed straight line becomes known.

Draw a straight line parallel to  $BC$ , and at a distance from it equal to that just determined. With  $A$  as a centre, and radius equal to that already determined describe a circle. The intersection of the circle and this straight line will determine the position of  $O$ ; and then the positions of  $AP$  and  $AQ$  become known.

527. We shall first shew that a certain straight line can be found the tangents drawn from any point of which to the two circles are equal: then the intersection of this straight line with the given straight line determines the required point.

Let  $A$  and  $B$  be the centres of the given circles. Draw any circle intersecting the two given circles; let  $EF$  be the common chord of the first circle and the third circle; let  $CD$  be the common chord of the second circle and the third circle. Let  $FE$  and  $DC$  produced meet at  $G$ ; from  $G$  draw  $GK$  perpendicular to  $AB$ . Then the tangents drawn to the two given circles from any point in  $GK$  will be equal. For take any point  $P$  in that straight line. Also from  $G$  draw  $GS$  to touch the first given circle, and  $GT$  to touch the second. The rectangle  $GE$ ,  $GF$  is equal to the rectangle  $GC$ ,  $GD$ , by III. 36 Corollary. Therefore the square on  $GS$  is equal to the square on  $GT$ , by III. 36; that is the excess of the square on  $GA$  over the square on  $SA$  is equal to the excess of the square on  $GB$  over the square on  $TB$ , by I. 47. Therefore the squares on  $GK$  and  $KA$  diminished by the square on  $SA$  are

equal to the squares on  $GK$  and  $KB$  diminished by the square on  $TB$ , by I. 47. Therefore the excess of the square on  $KA$  over the square on  $SA$  is equal to the excess of the square on  $KB$  over the square on  $TB$ . Add the square on  $PK$  to both these equals; and then we find that the excess of the square on  $PA$  over the square on  $SA$  is equal to the excess of the square on  $PB$  over the square on  $TB$ . Thus the square on the tangent from  $P$  to one circle is equal to the square on the tangent from  $P$  to the other circle.

Therefore the intersection of  $GK$  with the given straight line is the point required.

528. Let  $AB$  be the fixed chord,  $CD$  the other chord. Let  $AD$  and  $BC$  intersect at  $O$ . The angle  $AOB$  is measured by half the sum of the arcs  $AB$  and  $CD$ , by page 294 of the *Euclid*, and is therefore constant. Hence the locus of  $O$  is a certain segment of a circle, described on  $AB$ , by page 276 of the *Euclid*.

529. The points  $C, A, B$ , are in one straight line; so are  $A, E, D$ , and also  $B, D, F$ ; III. 11 and III. 12. Join  $CE$ , and produce it to meet  $BF$  at  $H$ . The angle  $BHC$  is equal to the angles  $ADB$  and  $DEH$ , and also equal to the angles  $BFC$  and  $HCF$ , by I. 32; therefore the angles  $ADB$  and  $DEH$  are together equal to the angles  $HCF$  and  $BFC$  together. But the angle  $BFC$  is equal to the angle  $BCF$ , by I. 5; and the angle  $DEH$  is equal to the angle  $BCE$ , by I. 15 and I. 5. Therefore the angles  $ADB$  and  $BCE$  are together equal to the angles  $BCF$  and  $HCF$  together; that is to the angle  $BCE$  and twice the angle  $ECF$ . Take away the common angle  $BCE$ ; then the angle  $ADB$  is equal to twice the angle  $ECF$ .

530. By *Euclid*, page 269, the square described on the sum of  $AP$  and  $CP$  together with the square described on the difference of  $AP$  and  $CP$  is equal to twice the sum of the squares on  $AP$  and  $CP$ , that is to twice the square on  $AC$ , by I. 47. Hence the square described on the sum of  $AP$  and  $CP$  is always less than twice the square on  $AC$ , except when  $AP$  and  $CP$  are equal. Therefore the greatest value of the sum of  $AP$  and  $CP$  occurs when  $CP$  is equal to  $AP$ .

531. The angle  $BPQ$  is measured by half the sum of the arcs  $AL$  and  $BM$ , by page 294 of the *Euclid*; similarly the angle  $BQP$  is measured by half the sum of the arcs  $LB$  and  $CM$ . But the arc  $AL$  is equal to the arc  $BL$ , and the arc  $BM$  is equal to the arc  $CM$ . Therefore the angle  $BPQ$  is equal to the angle  $BQP$ .

Again, let  $MN$  cut  $BC$  at  $R$  and  $CD$  at  $S$ , then in like manner  $CRS$  is an isosceles triangle. Now by I. 32 the angles of the two triangles  $BPQ$ ,  $CRS$ , are together double the angles of the triangle  $MQR$ ; and  $BPQ$  and  $BQP$  are together double of  $MQR$ , by I. 15; so also  $CRS$ , and  $CSR$  are together double of  $MRQ$ ; therefore the two angles  $ABC$ , and  $BCD$  are together double of the angle  $LMN$ .

532. Let  $AB$  be the given chord of the circle. Suppose  $C$  to be the required point on the circumference; join  $CA$  and  $CB$  and from  $AC$  cut off  $AD$  equal to the given difference. Join  $DB$ . Then  $CDB$  is an isosceles triangle; and as the angle  $C$ , being that subtended by a known chord, is known the angle  $CDB$  is known; and therefore also the angle  $ADB$  is known.

Hence the point  $D$  can be found. For it is on a segment of a circle described on  $AB$  containing a known angle; and it is on a circle described

from  $A$  as centre with a known radius. When  $D$  is known, by joining  $AD$  and producing it to meet the given circle  $C$  can be found.

533. Let  $AB$  denote the base of the triangle, and  $CD$  the perpendicular from the vertex on the base. Suppose  $DA$  greater than  $DB$ , and from  $DA$  cut off  $DE$  equal to  $DB$ , and join  $EC$ . Then we have given  $AE$  which is the difference of  $DA$  and  $DB$ ; also the sum of  $CA$  and  $CE$ , for this is equal to the sum of  $CA$  and  $CB$ ; and the angle  $ACE$ , for this is the difference of the angles  $CED$  and  $CAB$ , by I. 32, that is the difference of the angles  $CBD$  and  $CAB$ . Hence the triangle  $ACE$  can be constructed by Exercise 268; and then the triangle  $ACB$ .

534. Suppose the segment  $APB$  to fall within the segment  $AQB$ . Let  $AT$  within the segment  $AQB$  be a tangent at  $A$  to  $APB$ , and produce  $TA$  to  $R$ . Let  $AS$  be a tangent at  $A$  to  $AQB$ .

Then the angle  $PAT$  is equal to the angle  $PBA$ , and the angle  $BAS$  is equal to the angle  $BQA$ ; by III. 32. The angle  $PAR$  will be equal to the sum of the angles  $PAB$  and  $APB$ , by I. 13, III. 32 and I. 32. That is the angles  $PAB$ ,  $BAS$ ,  $SAR$  are together equal to the angles  $PAB$ , and  $APB$ . Therefore the angles  $BAS$  and  $SAR$  together are equal to the angle  $APB$ , that is to  $PAQ$  and  $AQB$ . But the angle  $BAS$  was shewn to be equal to the angle  $AQB$ ; therefore the angle  $SAR$  is equal to the angle  $PAQ$ . That is the angle  $PAQ$  is equal to the angle between the tangents.

535. Let  $C$  be the centre of the given circle. Let  $T$  be the middle point of  $KL$ ,  $M$  the middle point of  $PQ$ , and  $N$  the middle point of  $RS$ . The angles  $CNA$ ,  $CTA$ ,  $CMA$  are all right angles by III. 8; therefore  $N$ ,  $T$ , and  $M$  are on a circle described on  $AC$  as diameter, by page 276 of the *Euclid*. The angle between  $MN$  and  $AL$  is measured by half the sum of the arcs  $AN$  and  $MT$ , by page 294 of the *Euclid*; that is by half the sum of the arcs  $AN$  and  $NT$ , by III. 26; that is by half the arc  $AT$ , which is a fixed arc. Thus the angle between  $MN$  and  $AL$  is constant, so that  $MN$  always remains parallel to itself.

536. Let  $EFGH$  be a quadrilateral, such that round it the quadrilateral  $ABCD$  can be described, so that the angle  $BEF$  is equal to the angle  $BFE$ , the angle  $CFG$  to the angle  $CGF$ , the angle  $DGH$  to the angle  $DHG$ , and the angle  $AHE$  to the angle  $AEH$ . Then a circle may be described about the quadrilateral  $EFGH$ .

The angles  $HEF$ ,  $AEH$ ,  $BEF$  are together equal to two right angles, by I. 13; and so are the angles  $HGF$ ,  $DGH$ ,  $CGF$ . Therefore the angles  $HEF$ ,  $AEH$ ,  $BEF$ ,  $HGF$ ,  $DGH$ ,  $CGF$  are together equal to four right angles. Now in the four triangles  $AEH$ ,  $BFE$ ,  $FCG$ , and  $GDH$  the sum of all the angles is eight right angles; also the sum of the four angles at  $A$ ,  $B$ ,  $C$ ,  $D$  is four right angles; therefore the sum of the remaining angles is four right angles. Hence we find that the sum of the four angles  $AEH$ ,  $BEF$ ,  $DGH$ ,  $CGF$  is two right angles. Therefore the angles  $HEF$  and  $HGF$  are together equal to two right angles; and therefore a circle would go round  $HEFG$ , by page 276 of the *Euclid*.

537. From  $A$  draw the straight line  $ABC$  passing through the centres of the two circles, meeting the inner circle at  $B$ , and the outer circle at  $C$ . Suppose that the straight line  $AED$  is such as is required, meeting the inner

circle at  $E$ , and the outer circle at  $D$ , and making  $ED$  of the given length. Join  $CD$ ,  $EB$ , and from  $B$  draw  $BF$  perpendicular to  $CD$ .

The angles  $AEB$  and  $ADC$  are right angles, by III. 31; and the angle  $BFD$  is a right angle, by construction. Therefore  $EBFD$  is a rectangle, and  $BF$  is equal to  $ED$ .

Thus to solve the problem we describe a semicircle on  $BC$  and in it place the straight line  $BF$  equal to the given straight line, and then draw  $AED$  parallel to  $BF$ .

538.  $CD$  is parallel to  $AB$ , and therefore  $CD$  produced will cut  $AE$  at right angles, by I. 29; similarly  $AD$  produced will cut  $CE$  at right angles. Hence the perpendicular from  $E$  on  $AC$  will pass through  $D$ , by page 313 of the *Euclid*; that is if  $ED$  be joined and produced it will cut  $AC$  at right angles.

539. Let  $ABC$  be a triangle; let  $AD$ ,  $BE$ ,  $CF$  be the perpendiculars from the angular points on the opposite sides. By the *Euclid*, page 313, these perpendiculars meet at a point; denote the point by  $O$ .

Since the angles  $AFG$  and  $ADC$  are right angles a circle will go round  $AFDC$ , by the *Euclid*, page 276. Then the rectangle  $AO$ ,  $OD$  is equal to the rectangle  $CO$ ,  $OF$ , by III. 35. Similarly the rectangle  $AO$ ,  $OD$  is equal to the rectangle  $BO$ ,  $OE$ .

540. Let  $ABC$  be a triangle; let  $BF$  bisect the angle  $ABC$ , and meet  $AC$  at  $F$ ; let  $CG$  bisect the angle  $ACB$ , and meet  $AB$  at  $G$ . Let  $BF$  and  $CG$  intersect at  $O$ . From  $A$  draw  $AD$  perpendicular to  $BF$ , and  $AE$  perpendicular to  $CG$ . Then  $ED$  will be parallel to  $BC$ , and  $ED$ , produced if necessary, will bisect  $AC$  and  $AB$ .

The straight line  $AO$  will bisect the angle  $BAC$ , by page 312 of the *Euclid*. The angle  $OAF$  is half the angle  $BAC$ ; the angle  $AFB$  is equal to the angles  $FCB$ , and  $FBC$ , by I. 32; that is to the angle  $FCB$  and half the angle  $ABC$ ; therefore the angle  $AOF$  is equal to the difference of a right angle and half the angle  $ACB$ . But a circle would go round  $AEOD$ ; therefore the angle  $AED$  is equal to the angle  $AOF$ , that is to the difference of a right angle and half the angle  $ACB$ . But the angle  $AEC$  is a right angle; therefore the angle  $DEC$  is equal to half the angle  $ACB$ ; that is to the angle  $ECB$ . Therefore  $ED$  is parallel to  $BC$ , by I. 27.

Produce  $ED$  to meet  $AC$  at  $K$ . The angle  $KEC$  has been shewn equal to the angle  $KCE$ ; and  $AEC$  is a right angle; therefore the angle  $KEA$  is equal to the angle  $KAE$ , by I. 32; therefore  $KA$  and  $KC$  are each equal to  $EK$ , by I. 6; therefore  $KA$  is equal to  $KC$ .

Similarly  $DE$ , produced if necessary, will bisect  $AB$ .

541. Let  $AB$  be the diameter of the given circle. On  $AB$  describe a triangle  $ABD$  equal to half the given rectilineal figure: see I. 45, Corollary. Through  $D$  draw a straight line parallel to  $AB$ , meeting the circle at  $C$ . Then the triangle  $ACB$  is equal to the triangle  $ADB$ , by I. 37; and is therefore equal to half the given rectilineal figure. Also the angle  $ACB$  is a right angle by III. 31.

From  $A$  draw the chord  $AE$  parallel to  $CB$ ; and join  $BE$ . Then the angle  $BAE$  is equal to the angle  $ABC$ , by I. 29; the right angle  $AEB$  is equal to the right angle  $BCA$ ; therefore the angle  $ABE$  is equal to the angle  $BAC$ . Thus the triangle  $ABE$  is equal to the triangle  $BAC$  in all respects; and therefore the figure  $ACBE$  is equal to the given rectilineal figure.

Also the angle  $CAE$  is a right angle; for it is equal to the two angles  $CAB$  and  $BAE$ ; that is to the two angles  $CAB$  and  $CBA$ , by I. 29; that is to a right angle. Similarly the angle  $CBE$  is a right angle. Therefore the figure  $ACBE$  is a rectangle.

542. Let  $O$  be the point of intersection of  $AD$  and  $BE$ ; join  $CO$  and produce it to meet  $AB$  at  $L$ . Then  $CL$  is perpendicular to  $AB$ , by page 313 of the *Euclid*. The two circles both pass through  $L$ , by III. 31; thus the rectangles  $HO$ ,  $OK$ , and  $CO$ ,  $OL$ , and  $FO$ ,  $OG$  are all equal. Therefore a circle will go through  $F$ ,  $G$ ,  $H$ ,  $K$  by page 277 of the *Euclid*.

543. Let  $AB$  be one diameter; and  $CD$  another diameter, at right angles to the former. Let  $AE$ ,  $CG$ ,  $BF$ ,  $DH$  be four parallel chords. Then the arcs  $EAG$ ,  $GDF$ ,  $FBH$ ,  $HCE$  will all be equal.

Join  $AC$ , the angle  $EAC$  is equal to the angle  $ACG$ , by I. 29; therefore the arc  $EC$  is equal to the arc  $AG$ , by III. 26. Therefore the arc  $EAG$  is equal to the arc  $AEC$ , that is to a quarter of the circumference. Similarly the arc  $HBH$  is a quarter of the circumference.

In like manner the arcs  $HCE$  and  $FDG$  are equal; and as they are together equal to half the circumference each of them is a quarter of the circumference.

544. From  $A$  draw  $AM$  perpendicular to  $EC$ , from  $B$  draw  $BN$  perpendicular to  $EC$ . Then we must shew that  $AM$  and  $BN$  are together equal to  $EC$ . For then the square described on  $EC$  will be equal to the rectangle  $EC$ ,  $AM$  together with the rectangle  $EC$ ,  $BN$ ; that is to twice the triangle  $AEC$  together with twice the triangle  $EBC$ ; that is to twice the figure  $AECB$ .

Through  $O$  the centre of the circle draw the diameter  $POQ$  parallel to  $EC$ ; let  $BN$  intersect  $PQ$  at  $R$ ; and from  $O$  draw  $OS$  perpendicular to  $EC$ . Then  $OS$  is equal to  $RN$ , by I. 34; therefore the sum of  $BN$  and  $AM$  is twice  $BR$ ; and  $EC$  is twice  $ES$ . Therefore we have to shew that  $ES$  is equal to  $BR$ .

Now the angle  $EDO$  is equal to the angle  $ROB$ , by I. 29; the right angle  $EOD$  is equal to the right angle  $BRO$ ; therefore the angle  $DEO$  is equal to the angle  $OBR$ .

In the two triangles  $EOS$  and  $BOR$  the angle  $OES$  is equal to the angle  $OBR$ , as just shewn; the right angles  $OSE$  and  $ORB$  are equal; and the side  $OE$  is equal to the side  $OB$ ; therefore  $ES$  is equal to  $BR$ , by I. 26.

545. Let  $O$  be the centre of the circle. Bisect  $BC$  at  $D$  and draw  $DE$  parallel to  $OC$ , meeting  $OB$  at  $E$ . Then  $E$  is the middle point of  $OB$ , and  $ED$  is half of  $OC$ : see Exercises 106 and 109. And  $D$  is the middle point of the diagonals of the parallelogram, by Exercise 78.

Thus the required locus is the circle having its centre at  $E$  and its radius equal to half  $OC$ , that is to half the radius of the given circle.

546. Describe the circle which is obtained in the solution of Exercise 545 for the required locus. Join  $AE$  and produce it to meet the circumference of this circle at  $H$ . Then  $AH$  is the required direction in order that the diagonal may have its greatest possible length; and its length is twice  $AH$ .

547. Let  $A$  and  $B$  be the centres of the circles. Suppose that  $AD$  touches the circle having its centre at  $B$ , and that  $BC$  touches the circle having its centre at  $A$ , where  $C$  and  $D$  are on opposite sides of  $AB$ . Let

$AD$  cut at  $K$  the circle which has its centre at  $A$ ; and let  $BC$  cut at  $H$  the circle which has its centre at  $B$ . Join  $AC$ ,  $BD$ ,  $HK$ .

Then  $AC$  is equal to  $BD$ , and  $CB$  is equal to  $AD$ , by supposition; and the angles at  $C$  and  $D$  are right angles. Therefore  $ACBD$  is a rectangle. Also  $CB$  is equal to twice  $BH$ , and  $AD$  is equal to twice  $AK$  by supposition. Hence  $CHKA$  is a square, and  $HK$  touches both circles, and is equal to  $AC$ .

548. Let  $A$  and  $B$  be the centres of the two circles. From  $A$  as centre, with radius equal to the diameter of either of the given circles, describe a circle cutting at  $P$  the circle which has its centre at  $B$ . From  $P$  draw two tangents  $PT$ ,  $PS$  to the circle which has its centre at  $A$ . Produce  $TP$  to meet at  $K$  the circle which has its centre at  $B$ ; and produce  $SP$  to meet this circle at  $L$ . Join  $BL$ ,  $BK$ .

The angle  $TAP$  is equal to the angle of an equilateral triangle; this may be shewn by producing  $AT$  to  $R$ , so that  $RT$  may be equal to  $TA$ , and joining  $PR$ . Similarly the angle  $SAP$  is equal to the angle of an equilateral triangle. Thus the angles  $TPA$  and  $SPA$  are each equal to half the angle of an equilateral triangle; and therefore the angle  $LPK$ , that is the angle  $TPS$ , is equal to the angle of an equilateral triangle. Hence the angle  $LBK$  is twice the angle of an equilateral triangle.

In the triangles  $TAS$  and  $LBK$  the sides  $TA$ ,  $SA$  are equal to the sides  $LB$ ,  $KB$  each to each, and the angle  $TAS$  is equal to the angle  $LBK$ ; therefore  $TS$  is equal to  $LK$ .

It may happen that  $TP$  or  $SP$  cuts the circle which has its centre at  $B$ , and so does not require to be produced. The demonstration is not essentially changed.

It is necessary that the distance between  $A$  and  $B$  should not be greater than three times the radius, in order that the preceding solution may hold. There are two solutions if the described circle cuts the circle which has its centre at  $B$ , one if these circles touch, and none if one circle falls without the other.

549. The angle  $BCG$  is a right angle, by construction; the angle  $BFG$  is a right angle, by III. 31: therefore a circle will go round  $BFGC$ . Therefore the rectangle  $BA$ ,  $AC$  is equal to the rectangle  $FA$ ,  $AG$ .

Again, if a circle be described on  $BD$  as diameter it will pass through  $C$ , and  $AD$  will be a tangent to it because the angle  $ADB$  is a right angle. Therefore the rectangle  $BA$ ,  $AC$  will be equal to the square on  $AD$ , by III. 36.

550. Let  $AB$  be the given base. On  $AB$  describe a segment of a circle  $ACB$  containing an angle equal to the given vertical angle. Bisect  $AB$  at  $E$ ; through  $E$  draw  $ED$  at right angles to  $AB$ , meeting the segment at  $D$ ; and complete the circle. Produce  $DE$  to meet the circle again at  $F$ .

Suppose  $C$  the required vertex of the triangle: join  $FC$ , cutting  $AB$  at  $G$ . Then, as in Exercise 549, the rectangle  $FG$ ,  $FC$  is equal to the square on  $FB$ , that is to a known quantity. And  $FC$  bisects the angle  $ACB$ , so that  $GC$  is known by enposition; then  $FC$  may be found by the aid of Exercise 502. From  $F$  as centre with the radius just determined describe a circle; then the intersection of this with the segment  $ADB$  will determine the point  $C$ .

551. The angle  $APB$  is measured by the half the sum of the arcs  $AB$  and  $DE$ , and is therefore a known angle. Similarly the angle  $BPC$  is known. Join  $AB$ , and on it describe a segment of a circle containing an angle equal to



the former; join  $BC$ , and on it describe a segment containing an angle equal to the latter. Then the intersection of the two segments will determine the point  $P$ .

552. Let  $A$  be any point on the circumference of the circle. From  $A$  draw  $AB$  perpendicular to  $OB$  one of the given straight lines, and  $AC$  perpendicular to  $OC$  the other given straight line. Also through  $A$  draw a straight line meeting  $OB$  at  $E$ , and  $OC$  at  $D$ ; and equally inclined to the two given straight lines.

Then the angle  $OED$  is equal to the angle  $ODE$  by construction, and therefore the angle  $CAD$  is equal to the angle  $CDA$ , by I. 29; therefore  $AC$  is equal to  $DC$ . Hence the sum of  $AB$  and  $AC$  is equal to the sum of  $OC$  and  $CD$ , that is equal to  $OD$ .

Thus if the straight line  $DA$  cuts the circle we can obtain a *less* value of the sum of  $AB$  and  $AC$  by taking instead of  $A$  some point on the arc between  $DE$  and  $O$ . In this way we see that when the sum of  $AB$  and  $AC$  is least the point  $A$  must be such that the *tangent* to the circle there is equally inclined to the fixed straight lines, and is *between*  $O$  and the circle.

Similarly in order that the sum of  $AB$  and  $AC$  may be greatest the tangent at  $A$  must be equally inclined to the fixed straight lines, and the circle and the point  $O$  be on the *same* side of the tangent.

553. Let the segments described on  $AC$  and  $AB$  intersect at  $D$ . Then the angles  $ADB$  and  $ACB$  are together equal to two right angles, so are the angles  $ADC$  and  $ABC$ ; therefore the angles  $BDC$  and  $BAC$  are together equal to two right angles by I. 15, Corollary 2. Therefore the segment described on  $BC$  will pass through  $D$ : thus the segments all pass through one point.

The angles  $ADB$  and  $ACB$  are together equal to two right angles; therefore the angle  $ACB$  is equal to the angle contained by the remaining part of the circle of which  $ADB$  is a segment, by III. 22. Hence it follows that the circle of which  $ADB$  is a segment is equal to the circle which could be described round the triangle  $ABC$ . Similarly this holds for the circles of which  $ADC$  and  $BDC$  are segments. Hence the three circles are equal. Produce  $BD$  to meet  $AC$  at  $M$ , and produce  $CD$  to meet  $AB$  at  $N$ . Since the circles  $ADB$  and  $ADC$  are equal the angles  $NBD$  and  $MCD$  are equal, by III. 28 and III. 27; the angles  $NDB$  and  $MDC$  are equal, by I. 15; therefore the angle  $DNB$  is equal to the angle  $DMC$ , by I. 32; therefore the angle  $AND$  is equal to the angle  $AMD$ . The angles  $NDM$  and  $NAM$  are together equal to the angles  $BDC$  and  $NAM$ , that is to two right angles. Hence the angles  $AND$  and  $AMD$  are together equal to two right angles; and as they are equal each of them is a right angle. Similarly  $AD$  produced meets  $BC$  at right angles.

#### IV. 1 to 16.

554. The three perpendiculars meet at a point, by page 313 of the *Euclid*: denote this point by  $O$ . In the triangles  $AFC$  and  $OEC$  the angle at  $C$  is common; the right angle  $AFC$  is equal to the right angle  $OEC$ : therefore the angle  $FAC$  is equal to the angle  $EOC$ .

Since the angles  $BEC$  and  $ADC$  are right angles a circle will go round  $OECD$ , by the *Euclid*, page 276; therefore the angle  $EDC$  is equal to the angle  $EOC$ ; therefore the angle  $EDC$  is equal to the angle  $BAC$ .

Similarly the angle  $FDB$  is equal to the angle  $BAC$ . Therefore  $DE$  and  $DF$  are equally inclined to  $BC$ , and therefore to  $AD$ .

555. Let  $ABC$  be a triangle; suppose that the inscribed circle touches  $BC$  at  $D$ ,  $CA$  at  $E$ , and  $AB$  at  $F$ . Construct the triangle  $FDE$ ; from  $D$  draw  $DP$  perpendicular to  $EF$ , from  $E$  draw  $EQ$  perpendicular to  $FD$ , and from  $F$  draw  $FR$  perpendicular to  $ED$ . Then will  $PQ$ ,  $QR$ ,  $RP$  be parallel to  $AB$ ,  $BC$ ,  $CA$  respectively.

$AB$  touches the inscribed circle  $DEF$ ; therefore the angle  $AFP$  is by III. 32 equal to the angle  $FDE$ . But in the solution of Exercise 554 it is shewn that the angle  $FPQ$  is equal to the angle  $FDE$ . Therefore the angle  $FPQ$  is equal to the angle  $AFP$ ; therefore  $PQ$  is parallel to  $AB$ , by I. 27.

Similarly  $QR$  is parallel to  $BC$ , and  $PR$  is parallel to  $CA$ .

556. Draw the circumscribing circle; then as one angle is given the side opposite this angle is given in magnitude. For if from any point on the circumference of the circumscribing circle we draw two straight lines containing an angle equal to the given angle, the chord which they intercept will be equal in magnitude to the side. Thus the problem is reduced to that of Exercise 293.

557. Let  $AB$  be the base; then the vertices of all the triangles lie on a segment of a circle described on  $AB$ ; see the *Euclid* p. 276.

Let  $ACB$  be one of the triangles. Produce  $AB$ ,  $AC$  to  $H$  and  $K$ , and bisect the angles  $KCB$ ,  $CBH$ , by the straight lines  $CE$ ,  $BE$  meeting at  $E$ . Then as in Exercise 282,  $E$  is the centre of the circle touching  $BC$  and  $AB$ ,  $AC$  produced; and the straight line  $AE$  bisects the angle  $CAB$  as in Exercise 280.

Since the angle  $AEB$  is the difference of the angles  $EBH$  and  $EAB$  by I. 82; and  $CBH$  is double of  $EBH$ , and  $CAB$  double of  $EAB$ , therefore the angle  $ACB$  is double of the angle  $AEB$ , and the point  $E$  lies on the segment of a circle described on  $AB$  and containing an angle  $AEB$  equal to half the given angle  $ACB$ . Let this segment  $AEB$  be described.

Bisect  $AB$  at  $G$ , and draw  $GDF$  at right angles to  $AB$  meeting the segment  $ACB$  at  $D$ , and the segment  $AEB$  at  $F$ , and join  $AD$ ,  $DB$ ,  $FA$ ,  $FB$ . Then the angle  $ADB$  is double of the angle  $AFB$  by construction; also by I. 4 these angles are bisected by  $GF$ , therefore the angle  $ADG$  is double of  $AFD$  and  $AD$  is equal to  $DF$ . Similarly  $DB$  is equal to  $DF$ , and therefore  $D$  is the centre of the segment  $AEB$ , by III. 9.

558. Let  $AD$  meet  $BC$  at  $M$ . The angles  $ACM$  and  $CAM$  are together equal to a right angle; the angles  $ACB$  and  $CBE$  are together equal to a right angle; therefore the angle  $CAD$  is equal to the angle  $CBE$ . But the angle  $CAD$  is equal to the angle  $CBD$ , by III. 21. Therefore the angle  $MBD$  is equal to the angle  $MBL$ . Also the right angle  $DMB$  is equal to the right angle  $LMB$ ; and  $BM$  is common to the two triangles  $BMD$  and  $BML$ : therefore  $MD$  is equal to  $ML$ , by I. 26.

559. A circle may be described round the regular pentagon; then the angle  $DAC$  standing on  $DC$ , is equal to the angle  $ADB$  standing on  $AB$ , which is equal to  $DC$ . Therefore  $AO$  is equal to  $OD$ , by I. 6.

Again, the angle  $COD$  is equal to the sum of the angles  $CBO$  and  $BCO$ , that is to twice the angle  $CBO$ ; and the angle  $ACD$  is equal to the sum of the angles  $ACE$  and  $ECD$ , that is to twice the angle  $ACE$ : therefore the

angle  $COD$  is equal to the angle  $OCD$ ; therefore  $OD$  is equal to  $DC$ ; and therefore  $OA$  is equal to  $CD$ .

Thus the triangle  $ACD$  is exactly like the triangle  $ABD$  of IV. 10; the sides  $AC$  and  $AD$  are equal; the angles  $ADC$  and  $ACD$  are each double of the angle  $DAC$ ; and  $AO$  is equal to  $DC$ . Therefore as in IV. 10 we had the rectangle  $AB, BC$  equal to the square on  $BD$ , so here we have the rectangle  $AC, CO$  equal to the square on  $DC$ .

560. The angles  $CQR$  and  $CPR$  are right angles; therefore a circle described on  $CR$  as diameter will go round  $CQRP$ . This circle will be of constant magnitude; for  $PQ$  is of constant length, and the angle  $PCQ$  is constant. Thus the distance of  $R$  from  $C$  is constant.

Again, let  $PM$  be perpendicular to  $CQ$ , and  $QN$  perpendicular to  $CP$ . Then a circle will go round  $QMNP$ ; and this circle will be of constant magnitude, for its diameter  $PQ$  is of constant magnitude. Also  $MN$  will be of constant magnitude, for the angle  $MQN$  is the difference between a right angle and the angle  $C$ , and is therefore constant.

Finally a circle described on  $CS$  as diameter will go round  $CMSN$ ; and it will be of constant magnitude; for  $MN$  is of constant length, and the angle  $MCN$  is constant. Thus the distance of  $S$  from  $C$  is constant.

561. Let  $AB$  be the hypotenuse,  $ACB$  one of the right-angled triangles. It follows from the solution of Exercise 293 that the required locus is a segment of a circle  $ADB$  containing an angle equal to a right angle and a half. Complete the circle of which this segment is part; let  $O$  be the centre. Then the angle contained in the remaining part of the circle is half a right angle, by III. 22; therefore the angle  $AOB$  is a right angle, by III. 20. Therefore the arc  $ADB$  is a quarter of the circumference.

562.  $D$  is the centre of the circle described round  $ABC$ , by IV. 5; therefore the locus of  $D$  is the straight line which bisects  $AB$  at right angles: see III. 1.

563. Let  $AB$  be the given base; let  $ACB$  represent the triangle,  $O$  the centre of the inscribed circle,  $P$  the centre of the circle which touches  $AB$ , and touches  $CA$  and  $CB$  produced. Then  $OP$  is a known length. Suppose the angle  $CAB$  given.

$OA$  bisects the angle  $CAB$ , and  $PA$  bisects the angle between  $AB$  and  $CA$  produced; therefore  $OAP$  is a right angle. Similarly  $OBP$  is a right angle. A circle may be described on the known length  $OP$  as a diameter. Then the angle  $AOB$  subtended in such a circle by a known chord  $AB$  becomes known; also the angle  $OAB$  is known, for it is half the angle  $CAB$ . Thus all the angles of the triangle  $OAB$  are known, and the triangle can be constructed. Then make the angle  $CAB$  equal to twice the angle  $OAB$ , and the angle  $CBA$  equal to twice the angle  $OBA$ : thus we obtain the required triangle  $ACB$ .

564. Let  $A$  be the given point in the given straight line  $AE$ ; then the centre of the required circle must be on the straight line drawn through  $A$  at right angles to  $AE$ . Let  $C$  be the centre of the given circle, and suppose the required circle to cut it at the points  $L$  and  $M$ : then by supposition  $L, C$ , and  $M$  are in one straight line; suppose  $AC$  produced to meet the required circle again at  $D$ : then the rectangle  $AC, CD$  is equal to the rectangle  $LC, CM$ , by III. 35; that is to the square on  $LC$ . Now as  $LC$  and  $AC$  are

known this determines  $CD$ ; thus  $D$ , a point through which the required circle is to pass, is known, and the required centre must be on the straight line which bisects  $AD$  at right angles. The centre of the required circle is therefore at the point of intersection of two known straight lines, and therefore its position is determined.

565. Let  $A$  be the given point,  $B$  the centre of one of the given circles,  $C$  the centre of the other given circle. Join  $AB$  and produce it to a point  $H$  such that the rectangle  $AB, BH$  is equal to the square on the radius of the circle which has its centre at  $B$ ; then by the solution of Exercise 564 we see that  $H$  is a point through which the required circle must pass. Similarly join  $AC$  and produce it to a point  $K$  such that the rectangle  $AC, CK$  is equal to the square on the radius of the circle which has its centre at  $C$ ; then  $K$  is a point through which the required circle must pass. Thus we have only to describe a circle round the triangle  $AHK$ , which we do by IV. 5.

566. Describe an equilateral triangle  $ABC$  in the given circle. Bisect the sides  $BC, CA, AB$  at the points  $D, E, F$ , respectively. From the points  $D, E, F$ , draw  $DO, EO, FO$  at right angles to the straight lines  $BC, CA, AB$  respectively meeting one another at the point  $O$ . Describe a circle in the triangle formed by the tangent at  $A$  and the straight lines  $OE, OF$  produced. This will be one of the three required circles. Similarly the other two required circles can be drawn.

567. This can be shewn from IV. 10. There the straight line  $ACB$  is divided at  $C$  in the manner of II. 11. Also  $BD$  is equal to  $AC$ . The angle  $BAD$  is one-fifth of two right angles, that is one-tenth of four right angles. This is the angle subtended at the centre of a circle by the side of a regular decagon inscribed in the circle: therefore  $BD$  is the side of a regular decagon inscribed in the circle.

568. Let  $ABCDE$  be a regular pentagon inscribed in a circle. Let  $F$  be the centre of the circle. From any angle  $D$  draw the diameter  $DH$  cutting the straight line  $AB$  at the point  $I$ . This will be perpendicular to the opposite side  $AB$ . Join  $DA, DB$ . Then the triangle  $ADB$  is isosceles; and  $DH$  will bisect  $AB$ . Join  $FA, AH$ ; then  $AH$  will be the side of a regular decagon inscribed in the circle.

Now the angle  $AFH$  is equal to the angle  $ADB$  by III. 20, therefore the isosceles triangle  $AFH$  is equiangular to the isosceles triangle  $ADB$  by I. 32, and the angle  $FAH$  is double of the angle  $AFH$ . But by Exercise 567 the side  $AH$  is equal to the greater segment of the radius  $HF$  cut as in II. 11, therefore if the radius  $FH$  be cut at  $G$  so that the rectangle  $FH, HG$  is equal to the square on  $FG$ , then  $AH$  is equal to  $FG$ .

Join  $AG$ . The square on  $AB$  is equal to four times the square on  $AI$ . Therefore the square on  $AB$  together with four times the square on  $IH$  is equal to four times the square on  $AH$ . But as in IV. 10 we have  $AG$  equal to  $AH$ , therefore  $GH$  is bisected at  $I$ , therefore the square on  $GH$  is equal to four times the square on  $HI$ : therefore the square on  $AB$  together with the square on  $GH$  is equal to four times the square on  $FG$ . But since the square on  $FG$  is equal to the rectangle  $FH, HG$ , the sum of the squares on  $FH, HG$  is equal to three times the square on  $FG$  by II. 7. Therefore the square on  $AB$  is equal to the squares on  $FH, FG$ .

569. Let  $ABC$  be the given triangle,  $B$  the vertex. Describe a circle about it. Let  $F$  be the centre of this circle. Join  $FB$ . On  $FB$  as diameter describe a circle cutting  $AC$  at  $D$ . Join  $BD$ . This shall be the required straight line. Produce  $BD$  to meet the circumference of the circumscribed circle at  $E$ . Join  $FD$ . Then  $FDB$  is a right angle by III. 31, therefore  $ED$  is equal to  $DB$  by III. 8. The rectangle  $AD, DC$  is equal to the rectangle  $ED, DB$  by III. 35, that is to the square on  $DB$ .

570. Draw a quadrilateral  $DBEF$  and let  $BD, EF$  produced intersect at  $A$ , and let  $BE, DF$  produced intersect at  $C$ . Describe a circle about the triangle  $AEB$  and another about the triangle  $BDC$ . Let  $BO$  be the common chord of the circles. Join  $AO$  and  $DO$ . Then the angles  $AOB$  and  $AEB$  are equal by III. 21; and the angle  $AOB$  is equal to the sum of the angles  $AOD$  and  $DOB$ ; also the angle  $AEB$  is equal to the sum of the angles  $BCD$  and  $DFA$  by I. 32 and I. 15. Therefore the angle  $AEB$  is equal to the sum of the angles  $DOB$  and  $DFA$  by III. 21.

Therefore the angle  $AOD$  is equal to the angle  $DFA$ . Therefore the circle described about the triangle  $AFD$  passes through the point  $O$  by page 276 of the *Euclid*.

Similarly it may be shewn that the circle which circumscribes  $EFC$  also passes through  $O$ .

571. Let  $ABC$  be a triangle. From  $A$  draw a perpendicular on  $BC$ . From  $B$  draw a perpendicular on  $AC$ ; let these perpendiculars cut at  $D$ . From  $C$  draw  $CG$  perpendicular to  $AB$ . This will pass through  $D$  by p. 313 of the *Euclid*.

The rectangle  $GA, GE$  is equal to the rectangle  $GC, GD$  and therefore equal to the rectangle  $GB, GF$  by III. 86 Corollary.

The angle  $DAG$  is equal to the angle  $ECG$  by III. 21. The angle  $DAG$  is also equal to the angle  $BCG$  by I. 32. Therefore the angle  $ECG$  is equal to the angle  $BCG$ . Therefore the triangles  $ECG$  and  $BCG$  are similar, and  $CG$  is common, therefore  $EG$  is equal to  $BG$  by I. 26. Now the rectangle  $GA, GE$  is equal to the rectangle  $GB, GF$ , therefore  $GA$  is equal to  $GF$ , therefore  $AE$  is equal to  $FB$ .

572. Let  $O$  be the centre of the circle inscribed in the triangle  $ABC$  and let  $D, E, F$  be the centres of the escribed circles. Then  $AO$  is perpendicular to  $FE$ ; see Exercise 282. The result required follows as in Exercise 553.

573. Let  $ABDC$  be the quadrilateral. Let  $EA$  bisect the angle  $BAC$  and let  $EB$  bisect the angle  $ABD$ . Then  $E$  is the centre of the circle which touches internally the straight lines  $DB, BA, AC$ . Similarly let  $F$  be the centre of the circle which touches internally  $AC, CD, DB$ . In like manner let  $G, H$  be the centres of the other two circles. Then it can be easily shewn that the angles at  $E$  and  $F$  are together equal to two right angles. Therefore a circle can be described round the quadrilateral  $EFGH$  by page 276 of the *Euclid*. See Exercise 513.

574. Take the point  $P$  in the arc  $AB$ . Because  $DP, PF$  are respectively perpendicular to  $BC, BA$ , therefore the angle  $ABC$  is equal to the angle  $DPF$ , and the angle  $DPF$  is equal to the angle  $DEF$  by III. 21, therefore the angle  $ABC$  is equal to the angle  $DEF$ . Similarly the angle  $ACB$  is equal to the angle  $DFE$  and the angle  $BAC$  is equal to the angle  $EDF$ .

Therefore the sides of the triangle  $ABC$  are equal to the sides of the triangle  $DEF$  respectively by III. 26 and III. 29. Again, the arc  $BC$  is equal to the arc  $EF$ ; take away if necessary the common arc  $FC$ , then  $BE$  is parallel to  $CF$  by Exercise 219. Similarly it may be shewn that  $AD$  is parallel to either  $BE$  or  $CF$ .

575. We will take the case in which the point  $D$  falls within the given circle; the case in which  $D$  falls without the given circle can be treated in substantially the same manner. Let  $C$  be the centre of the described circle. Join  $QB, BC, CA, QC, CD$ . The two angles  $QBC, QAC$  are together equal to two right angles by III. 22. Now since  $CD$  is equal to  $CA$ , the angle  $CDA$  is equal to the angle  $CAD$ : therefore the two angles  $QDC, QAC$  are together equal to two right angles; therefore the angle  $QDC$  is equal to the angle  $QBC$ ; and the triangle  $DCB$  is equilateral, therefore the angle  $QBD$  is equal to the angle  $QDB$ , therefore the side  $QD$  is equal to the side  $QB$  by I. 6; therefore  $QC$  bisects the angle  $DCB$  by I. 8, therefore  $QB$  subtends at the centre of the given circle an angle equal to two-thirds of a right angle by III. 20, therefore  $QB$  is equal to the radius of this circle by IV. 15 Corollary, but  $QD$  has been shewn to be equal to  $QB$ . Therefore  $QD$  is equal to the radius of the circle.

576. Let  $ABCD$  be the given square, and  $P$  a point without it such that the angles  $APB, BPC, CPD$  are all equal. Then  $PAB$  will be greater than a right angle and  $PCB$  less than a right angle.

Draw  $BM, BN$  perpendiculars on  $PA, PC$  produced if necessary; then since  $BP$  bisects the angle  $APC$ ,  $BM$  will be equal to  $BN$ . Hence as  $BA$  is equal to  $BC$  the triangles  $BAM, BCN$  will be equal in every respect. Thus the angles  $BAP, BCP$  are together equal to two right angles, and therefore  $P$  lies on the circle described so as to pass through the points  $A, B, C$ ; that is the circle described about the square  $ABCD$ .

577. Let  $ABC$  be a triangle. Draw  $AP$  perpendicular to  $BC$ . Let the circle inscribed in the triangle  $APB$  touch  $AP, BP, AB$  at  $M, N, E$  respectively. Then the sum of the straight lines  $AP, BP$  is equal to the sum of the straight lines  $AM, MP, BN, NP$ , but  $AM$  is equal to  $AE, PM$  is equal to  $PN$  and  $BN$  is equal to  $BE$  by Exercise 176, therefore the sum of the straight lines  $AP$  and  $BP$  is equal to the sum of the straight lines  $AE, BE$  together with twice the straight line  $MP$ , that is equal to the sum of  $AB$  and twice the radius of the circle. Hence if  $X$  denote the diameter of the circle inscribed in  $ABP$ , the sum of  $X$  and  $AB$  is equal to the sum of  $AP$  and  $BP$ . Similarly if  $Y$  denote the diameter of the circle inscribed in  $ACP$ , the sum of  $Y$  and  $AC$  is equal to the sum of  $AP$  and  $CP$ . Therefore the sum of the straight lines  $X, Y, AB, AC$  is equal to the straight line  $BC$  together with twice the straight line  $AP$ . Two other results like this can be found; and then by addition the required result can be obtained.

578. Let  $O$  be the common centre. Take any point  $P$  on the circumference of the middle circle; join  $OP$  and produce it to  $Q$  making  $PQ$  equal to  $OP$ . With centre  $Q$  and radius equal to that of the smallest circle, describe a circle, and let one of the points at which it meets the outermost circle be  $S$ . Join  $QS$ ; from  $O$  draw  $OR$  a radius of the inner circle parallel to  $QS$ . Join  $SP, PR, QR, SO$ .

By construction  $SQ$  is equal to  $RO$  and  $QP$  is equal to  $PO$ . Then in the two triangles  $SPQ, RPO$  the two sides  $SQ, QP$  are equal to the two sides

$RO$ ,  $OP$  each to each, and the included angle  $SQP$  is equal to the included angle  $ROP$  by I. 29, therefore the base  $SP$  is equal to the base  $PR$  and the triangles are equal in all respects by I. 4, therefore the angle  $SPQ$  is equal to the angle  $OPR$ , therefore  $SP$ ,  $PR$  are in the same straight line; See I. 15.

## VI. 1 to D.

579. Draw  $PM$  perpendicular to  $AB$ . Then  $CD$  is to  $PM$  as  $AC$  is to  $AM$ , and  $CE$  is to  $PM$  as  $BC$  is to  $BM$  by VI. 4. Therefore the rectangle  $CD$ ,  $CE$  is to the square on  $PM$  as the rectangle  $AC$ ,  $BC$  is to the rectangle  $AM$ ,  $MB$ . But the square on  $PM$  is equal to the rectangle  $AM$ ,  $BM$  by III. 35, therefore the rectangle  $CD$ ,  $CE$  is equal to the rectangle  $AC$ ,  $BC$ , which is equal to the square on  $CF$  by III. 35. Therefore since the rectangle  $CD$ ,  $CE$  is equal to the square on  $CF$ ,  $CD$  is a third proportional to  $CE$  and  $CF$  by VI. 17.

580. From the middle point  $P$  of  $BD$  draw a straight line at right angles to  $BD$  meeting  $AB$  produced through  $C$  at the point  $O$ ; then  $OB$  is equal to  $OD$  by I. 4. Therefore the angle  $ODB$  is equal to the angle  $OBD$  by I. 5, which is equal to the sum of the angles  $OAD$  and  $ADB$  by I. 32. But the angle  $ADB$  is equal to the angle  $CDB$ , therefore the angle  $ODB$  is equal to the sum of the angles  $OAD$  and  $CDB$ , therefore the angle  $ODC$  is equal to the angle  $OAD$ . Also the angle  $AOD$  is common to the two triangles  $OCD$ ,  $ODA$ . Hence these triangles are similar by I. 32; therefore  $OD$  is to  $OC$  as  $AD$  is to  $DC$ , therefore  $OD$  is to  $OC$  as  $AB$  is to  $BC$  by VI. 3. But  $OD$  is equal to  $OB$ , therefore  $OB$  is to  $OC$  as  $AB$  is to  $BC$ , which shews that  $O$  is a fixed point. Hence the locus of  $D$  is a circle whose centre is  $O$  and radius  $OB$ .

581. Let  $ABCD$  be a square. Take  $BE$  equal to a fourth of  $BD$  and therefore equal to a third of  $DE$ . Join  $AE$  and produce it to meet  $BC$  at  $F$ . Then by the similar triangles  $ADE$  and  $BFE$  we have  $BF$  a third of  $AD$  or  $BC$ . Let the straight lines  $Bbc$ ,  $Ccd$ ,  $Dda$  be similarly drawn from  $B$ ,  $C$ ,  $D$ . Then by the symmetry of the construction it is evident that the figure  $abcd$  is both equilateral and equiangular, that is it is a square. And since the straight line  $BC$  is equal to three times the straight line  $BF$  the straight line  $Bc$  is equal to three times the straight line  $Bb$  and therefore the straight line  $bc$  is equal to twice the straight line  $Bb$ . And the square on  $BC$  is equal to the sum of the squares on  $Bc$ ,  $Cc$  by I. 47. But the square on  $Cc$  is equal to the square on  $Bb$ , therefore the square on  $BC$  is equal to the sum of the squares on  $Bc$ ,  $Bb$ , which is equal to ten times the square on  $Bb$ . Therefore the square  $abcd$  is equal to four times the square on  $Bb$  or four-tenths of the square on  $BC$  which is equal to two-fifths of  $ABCD$ .

582.  $AF$  bisects the angle  $A$  of the triangle by VI. 2, 3.

583. On the straight line  $AC$  describe a semicircle; from  $B$  draw  $BQ$  meeting the semicircle at  $Q$  so that the angle  $ABQ$  may be equal to half a right angle. From  $Q$  draw  $QP$  perpendicular to  $AC$ .

Since the angle  $PBQ$  is half a right angle, and the angle  $QPB$  is a right angle, therefore the remaining angle  $PQB$  is half a right angle, and is therefore equal to the angle  $ABQ$ ; therefore the side  $PB$  is equal to the side  $PQ$ , by I. 6. Therefore the square on  $PB$  is equal to the square on  $PQ$ , that is

to the rectangle  $AP, PC$ , by III. 35. Therefore  $PB$  is a mean proportional between  $PA$  and  $PC$ , by VI. 17.

584. The angle  $APB$  is constant and  $PA$  to  $PD$  is a known ratio, thus the angles  $PDA, PAD$  are constant by VI. 6. Then the angle  $PDA$  is equal to the angle  $PEB$  and the angle  $PAD$  is equal to the angle  $PBE$ . A circle will thus go round  $ABDE$ ; this is a fixed circle and  $DE$  subtends a constant angle in it, so that  $DE$  is of constant magnitude, and therefore at a fixed distance from its centre, that is  $DE$  always touches a fixed circle.

585. From  $A$  draw  $AF$  perpendicular to  $BC$ . Then the angle  $BAF$  is equal to two-thirds of a right angle by I. 26 and I. 32, therefore  $AF$  is half of  $AB$  as may be shewn by producing  $AF$  to double its length, therefore  $AF$  is to  $AB$  as  $DF$  is to  $DB$ . Therefore  $AD$  bisects the angle  $BAF$  by VI. 3, therefore the angle  $BAD$  is equal to the angle  $DBA$ , and the angle  $ADE$  is therefore equal to two-thirds of a right angle. Similarly also the angle  $DEA$ .

586. Let  $ABCD$  be a rectangle having the square on the side  $AB$  double of the square on the side  $BC$ ; also the square on the side  $DC$  double of the square on the side  $AD$ . Let  $AF$  be drawn from the angle at  $A$ , and  $CE$  from the angle at  $C$  perpendicular to the diagonal  $DB$ .

Now  $BE$  is to  $BC$  as  $BC$  is to  $BD$  by VI. 4. Therefore the rectangle  $BE, BD$  is equal to the square on  $BC$  by VI. 17, that is to one-third of the square on  $BD$  by I. 47, therefore  $BE$  is equal to one-third of  $BD$ . Similarly  $DF$  can be shewn to be equal to one-third of  $BD$ . Therefore the remainder  $FE$  is equal to a third of  $BD$ . Thus  $BE, EF, FD$  are all equal.

587. By IV. 4 the angles  $CAH, CAG$  are equal to one another, therefore the angle  $GAH$  is equal to the angle  $DAC$ ; and the isosceles triangles  $AGD, CAH$  are similar, therefore  $GA$  is to  $AH$  as  $DA$  is to  $AC$ ; thus the angle  $ADC$  is equal to the angle  $AGH$  by VI. 6, and  $GA$  is to  $GH$  as  $DA$  is to  $DC$ . Similarly it may be shewn that the angles  $BGK, BDC$  are equal to one another and that  $BD$  is to  $DC$  as  $GB$  is to  $GK$ . Again since  $DA$  is equal to  $DB$ , therefore  $GA$  is to  $GH$  as  $DB$  is to  $DC$ , therefore  $GA$  is to  $GH$  as  $GB$  is to  $GK$ . But  $AG$  is equal to  $GB$ , therefore  $GH$  is equal to  $GK$ .

588. Let  $ABC$  be a right-angled triangle and  $AB, BC$  the sides containing the right angle  $ABC$  on which are described the squares  $DB, BE$ . Let  $CD, AE$  be joined cutting the sides at  $H$  and  $G$ . Join  $DB, BE$  which form a straight line by I. 14, and  $HG$ . From the similar triangles  $DHA, BHC$   $HA$  is to  $HB$  as  $AD$  is to  $BC$ ; from the similar triangles  $BAG, CEG$ ,  $BG$  is to  $GC$  as  $BA$  is to  $CE$ . By comparing these two ratios  $HA$  is to  $HB$  as  $BG$  is to  $GC$ . But by VI. 4 from the similar triangles  $BAG, CEG$ ,  $BG$  is to  $GC$  as  $AG$  is to  $GE$ , therefore  $HA$  is to  $HB$  as  $AG$  is to  $GE$ , therefore  $HG$  is parallel to  $BE$  by VI. 2, therefore the angle  $BGH$  is equal to the angle  $GBE$  by I. 29, that is to half a right angle, and is therefore equal to the angle  $BHG$ , therefore  $BH$  is equal to  $BG$  by I. 6, therefore from the third mentioned proportion the rectangle  $AH, CG$  is equal to the square on  $BG$  by VI. 17, which is equal to the square on  $BH$ , therefore  $BG$  and  $BH$  are both mean proportionals between  $AH$  and  $CG$ .

589. Let  $OA, OB$  be the two given straight lines, and  $P$  the given point. The ratio of the straight lines from  $P$  is known and the angle they include.



Hence if the points where they meet the given straight lines be joined we have a triangle with known angles.

Make such a triangle  $pqr$ : on  $qr$  describe a segment of a circle having an angle equal to the angle  $O$ , and on  $pq$  a segment having an angle equal to the angle  $POB$ . Let  $o$  be the point where these segments meet. Join  $op$ . Then draw  $PQ$  making the angle  $OPQ$  equal to the angle  $opq$  and  $PR$  making the angle  $OPR$  equal to the angle  $opr$ .

590. Produce  $PE$  to meet the arc  $BO$  at  $G$ . Join  $AG$ .

Then the rectangle  $DE, EA$  is equal to the rectangle  $BE, EC$ , that is to the rectangle  $PE, EG$  by III. 35, therefore  $DE$  is to  $EP$  as  $GE$  is to  $EA$  by VI. 16; therefore the angle  $OAG$  is equal to the angle  $DPE$  by VI. 6. But the angle  $OAG$  is equal to twice the angle  $OPG$  by III. 20, therefore the angle  $DPE$  is bisected by the straight line  $OP$ , therefore the angles  $DPO, EPO$  are equal to one another.

591. Join  $CF$  and produce it to meet  $AB$  at  $H$ . Join  $HG$ .

Now  $HF$  is equal to  $FC$ ; see VI. 1; thus  $HG$  is equal and parallel to  $EC$  by I. 4, 27. Again the ratio of  $BO$  to  $BE$  is half that of  $BG$  to  $BE$ , that is half that of  $HG$  to  $AE$  by VI. 4, that is half that of  $HB$  to  $AB$  by VI. 2. Draw  $HK$  parallel to  $DC$ ; then  $DF$  is equal to  $FK$  by I. 26. Therefore the ratio of  $DF$  to  $DA$  is half that of  $DK$  to  $DA$ , that is half that of  $HB$  to  $AB$  by VI. 2. Therefore  $DF$  is to  $DA$  as  $BO$  is to  $BE$ .

592. Join  $AB, AF$ . First we shall shew that the square on  $AB$  is equal to the rectangle  $AE, AD$ . We have the square on  $AB$  equal to the square on  $AE$  together with the rectangle  $BE, EC$  by the note on III. 35 and III. 36 on page 277 of the *Euclid*. Also the rectangle  $BE, EC$  is equal to the rectangle  $AE, ED$  by III. 35, therefore the square on  $AB$  is equal to the square on  $AE$  together with the rectangle  $AE, ED$ , that is it is equal to the rectangle  $AE, AD$ . But  $AB$  is equal to  $AF$ : therefore the square on  $AF$  is equal to the rectangle  $AE, AD$ , therefore by the converse of VI. 8 we have the angle  $AEF$  a right angle. Similarly the angle  $AEG$  is a right angle, therefore  $GE, EF$  are in one straight line by I. 14.

593. The triangle  $ADE$  is equal to the triangle  $BFE$  by I. 38, therefore the triangle  $ADE$  together with the triangle  $ABE$  is equal to the triangle  $AFE$ ; therefore by I. 38 the triangle  $ADE$  together with the triangle  $BCG$  is equal to the triangle  $FCG$ . Take away the common part the triangle  $GHC$ ; therefore the triangle  $ADE$  together with the triangle  $BHC$  is equal to the triangle  $FHG$ .

594. Let  $CPQ$  be the straight line drawn from the angle  $C$  meeting the intersection of the two straight lines at  $P$  and the side  $AB$  at  $Q$ .

The triangle  $PEC$  is equal to one-third of the triangle  $PAE$  by VI. 1; also the triangle  $BEC$  is equal to one-third of the triangle  $BAE$ ; therefore the triangle  $BPA$  is equal to three times the triangle  $BPC$ , that is to twelve times the triangle  $BPD$ , therefore the straight line  $PA$  is equal to twelve times the straight line  $PD$ , therefore the triangle  $PDC$  is equal to one-twelfth of the triangle  $PAC$  by VI. 1, therefore the triangle  $BPD$  is equal to one thirty-sixth of the triangle  $PAC$ : therefore the triangle  $BPC$  is equal to one-ninth of the triangle  $PAC$ . Therefore this is also the ratio of the triangle  $BPQ$  to the triangle  $APQ$ , therefore it is the ratio of  $BQ$  to  $AQ$ .

595. The tangents so drawn are parallel to the sides of the inscribed figure, as may be shewn by drawing radii to the points of contact. Then the required result follows by Exercise 35.

596. Let  $ABC$ ,  $CBD$  be the two right-angled triangles, then the angle  $ACD$  is a right angle. Make  $BE$  equal to  $BC$ . Join  $EC$ : then  $BEC$  is the required triangle, for the three triangles are as  $AB$ ,  $BD$ ,  $EB$  by VI. 1, and  $EB$  is a mean proportional between  $AB$  and  $BD$  by construction and VI. 8 Corollary.

597. Join  $DE$ . Then  $DE$  is parallel to  $AB$  by VI. 2. From the similar triangles  $DCE$ ,  $CAB$ ,  $CD$  is to  $DE$  as  $CA$  is to  $AB$ , but  $CD$  is equal to one-third of  $CA$ , therefore  $ED$  is equal to one-third of  $AB$ . From the similar triangles  $OED$ ,  $BOA$ ,  $OE$  is to  $ED$  as  $OA$  is to  $BA$ , but  $ED$  is equal to one-third of  $BA$ , therefore  $OE$  is equal to one-third of  $OA$ , that is to one-fourth of  $EA$ . Similarly  $OD$  is equal to one-fourth of  $DB$ .

598. Let  $O$  and  $P$  be the centres of the two circles. Join  $CD$ ,  $PE$ . Then the angle  $CDA$  is a right angle by III. 31; also the angle  $PED$  is a right angle by III. 18, therefore  $PE$  is parallel to  $CD$  by I. 28, therefore  $AD$  is to  $DE$  as  $AC$  is to  $CP$  by VI. 2, therefore  $AD$  is to twice  $DE$  as  $AC$  is to twice  $CP$  by V. 4. Similarly  $BF$  is to twice  $FG$  as twice  $CP$  is to twice  $CO$ . Hence  $AD$  is to twice  $DE$ , as twice  $GF$  is to  $FB$ , by V. 11: therefore the rectangle  $AD$ ,  $FB$  is equal to four times the rectangle  $FG$ ,  $DE$  by VI. 16.

599. When circles cut at right angles the tangents at a point of intersection are at right angles to each other; and thus the radius of each circle is a tangent to the other circle at this point. Let  $D$  be the point where the circles which have  $B$  and  $C$  as centres meet; we have only to shew that the angle  $BDC$  is a right angle.

Since  $BD$  is equal to the tangent from  $B$  to the circle  $AC$ , the square on  $BD$  is equal to the rectangle  $BA$ ,  $BC$  by III. 36. Similarly the square on  $CD$  is equal to the rectangle  $CA$ ,  $CB$ . Therefore by II. 2 the square on  $BC$  is equal to the sum of the squares on  $BD$ ,  $DC$ , therefore the angle  $BDC$  is a right angle by I. 48.

600. Let  $BF$  cut  $AD$  at  $P$ . Then it may be shewn that the angle  $APB$  is a right angle and that the angle  $PAB$  is two-thirds of a right angle. Hence it follows that  $PA$  is one-half of  $AB$ . Also  $AD$  is twice  $AB$ : See IV. 15 Corollary. Hence  $AP$  is one-fourth of  $AD$  and therefore  $AP$  is one-third of  $PD$ .

601. Since the angle  $A$  is equal to the angle  $D$ , and  $AB$  is equal to  $DF$ , therefore the perpendicular from  $B$  on  $AC$  is equal to the perpendicular from  $F$  on  $DE$ . Therefore by VI. 1, the triangles are as  $AC$  to  $DE$ .

602. Let  $E$  be the centre of the inscribed  $D$  of the escribed circle, then  $B$ ,  $E$ ,  $D$ , are in one straight line. Draw  $EF$ ,  $DG$  perpendicular to  $BA$ . Then  $BE$  is to  $BD$  as  $FE$  is to  $DG$ , that is as  $EM$  is to  $DP$ , therefore  $PD$  is parallel to  $EM$  by VI. 7 and I. 28. And  $ND$  is parallel to  $EM$ . Therefore  $PD$  and  $DN$  must lie in a straight line.

603. The triangle  $DMA$  is similar to the triangle  $BAC$ ; therefore  $AM$  is to  $MD$  as  $CA$  is to  $AB$  by VI. 4. Therefore by V. 16,  $AM$  is to  $CA$  as  $MD$  is

to  $AB$ , that is as  $NA$  is to  $AB$ , therefore the triangle  $CAM$  is similar to the triangle  $BAN$ , by VI. 6, therefore the angle  $AMC$  is equal to the angle  $ANB$ , therefore the angle  $BMC$  is equal to the angle  $BNC$  by I. 13.

604. Let  $B$  be the middle point of the arc  $ABC$ . From  $B$  draw any two straight lines  $BF, BG$ , meeting the circumference at  $F$  and  $G$ , and the chord  $AC$  at  $D$  and  $E$ , respectively.

The angle  $BED$  is equal to the sum of the angles  $BOE, EBC$ , that is to the sum of the angles  $BFC, CFG$  by III. 27 and III. 21, that is to the angle  $BFG$ . Now  $BED$  and  $DEG$  are equal to two right angles; therefore  $BFG$  and  $DEG$  are equal to two right angles, therefore the points  $F, D, E, G$  are on the circumference of a circle. See page 276 of the *Euclid*.

605. Let  $H$  be the centre of the inscribed  $K$  of the escribed circle. The angle  $HAK$  is equal to the angle  $HBK$  that is to a right angle by IV. 4; therefore a circle would go round  $HAKB$ : see page 276 of the *Euclid*. Produce  $HD$  to meet the circumference of this circle again at  $L$ ; then the angle  $KLH$  is a right angle by III. 31; therefore  $DEKL$  is a rectangle and  $EK$  is equal to  $DL$ . And the rectangle  $AD, DB$  is equal to the rectangle  $HD, DL$ , that is to  $HD, EK$ . Similarly the rectangle  $AE, EB$  is equal to the rectangle  $HD, EK$ .

606. Let  $ABC$  be any triangle. Let  $DE$  be drawn parallel to the base  $BC$ . Let  $F$  be the middle point of  $DE$ . Join  $AF$  and produce it to meet  $BC$  at  $G$ .

Then  $AB$  is to  $AD$  as  $CA$  is to  $AE$  by VI. 2; but  $BA$  is to  $AD$  as  $BG$  is to  $DF$  and  $CA$  is to  $AE$  as  $CG$  is to  $FE$  by VI. 4, therefore  $DF$  is to  $FE$  as  $BG$  is to  $GC$ ; but  $DF$  is equal to  $FE$ , therefore  $BG$  is equal to  $GC$ . Hence the locus of  $F$  is the straight line drawn from  $A$  to the middle point of  $BC$ .

607. Let  $ABC$  be a triangle,  $AC$  the base and  $D$  the middle point of  $AC$ ; then  $BD$  bisects every straight line parallel to  $AC$ : see Exercise 606. Let  $EHLG$  be the parallelogram having the side  $HL$  in  $AC$ , and the sides  $EH, GL$  parallel to the fixed direction. Then  $F$ , the middle point of  $EG$ , is on  $BD$ . Draw  $FK$  parallel to the fixed direction; then  $P$  the middle point of  $FK$  is the intersection of the diagonals of the parallelogram  $EHLG$ . Now the straight line  $FK$  moves always parallel to itself and so the locus of its middle point is a fixed straight line passing through  $D$ .

608. The three bisectors meet at a point, see page 311 of the *Euclid*; the locus is a circle having its centre at the middle point of  $AB$  and its radius equal to one-sixth of  $AB$ . For let  $C$  be the right angle. Let the bisectors be  $BD, AE, CF$  and let them intersect at  $G$ . Because  $AF$  is equal to  $FB$ , the triangle  $AGF$  is equal to the triangle  $GFB$ . Again because  $AD$  is equal to  $DC$  the triangle  $ADB$  is equal to the triangle  $DCB$  and the triangle  $ADG$  is equal to the triangle  $DCG$  by I. 38, therefore the triangle  $AGB$  is equal to the triangle  $CGB$ , therefore the triangle  $FGB$  is equal to half the triangle  $CGB$ , therefore  $GF$  is equal to half  $GC$ , that is to one-third of  $FC$ . See VI. 1. But  $CF$  is equal to  $FA$  by Exercise 59, therefore the radius  $FG$  is equal to one-sixth of  $AB$ .

609. Draw a common tangent; let this meet the straight line which joins the centres at  $A$ . Then a straight line drawn through  $A$  and the

given point is the one required, for the parts intercepted by the circles cut off arcs bearing in both circles the same ratio to the whole circumference. See page 303 of the *Euclid*.

610. Let  $ABC$  be the given triangle and  $AC$  the base. With centre  $A$  and radius  $AC$  describe a circle; join  $B$  to the given point  $D$  and produce  $BD$  to meet the circumference of the circle at  $E$ . Join  $AE$ : draw  $DF$  parallel to  $AE$  meeting  $AB$  at  $F$ , and  $FG$  parallel to  $AC$  meeting  $BC$  at  $G$ . Then  $FD$  is to  $AE$  as  $BF$  is to  $BA$ , that is as  $FG$  is to  $AC$ , by VI. 4, but  $AE$  is equal to  $AC$ , therefore  $FD$  is equal to  $FG$ ; and  $DF$ ,  $FG$  are two adjacent sides of the required rhombus.

611. The triangle  $AFH$  is similar to the triangle  $EAC$ : for the angle  $FAH$  is equal to the angle  $BAC$  together with one right angle by IV. 9, that is to the angle  $EAC$ : and  $FA$  is to  $AH$  as  $EA$  is to  $AC$ , therefore the angle  $AFH$  is equal to the angle  $AEC$  by VI. 6. Similarly the angle  $BFG$  is equal to the angle  $CDB$  and the angle  $ECD$  is equal to the sum of the angles  $AEC$  and  $BDC$  as may be seen by drawing through  $C$  a straight line parallel to  $AE$ , therefore the angle  $ECD$  together with the angle  $HFG$  is equal to the angle  $AFB$  that is to a right angle by IV. 9.

*Miscellaneous.*

612. Draw  $OD$  perpendicular to the fixed straight line. In  $OD$  take a point  $E$  such that the rectangle  $OE$ ,  $OD$  may be equal to the rectangle  $OQ$ ,  $OP$ . Join  $EQ$ .

Since  $OE$  is to  $OQ$  as  $OP$  is to  $OD$  and since the triangles  $EOQ$ ,  $POD$  have a common angle  $DOP$ , they are similar by VI. 6; therefore the angle  $OQE$  is equal to the angle  $ODP$ , that is it is a right angle. Therefore the locus of  $Q$  is a circle having  $OE$  for its diameter. See page 276 of the *Euclid*.

613. Draw the diameter  $OD$ . In  $OD$  take a point  $E$  such that the rectangle  $OE$ ,  $OD$  may be equal to the rectangle  $OQ$ ,  $OP$ . Join  $EQ$ ,  $PD$ . Then as in Exercise 612 the triangles  $EOQ$ ,  $OPD$  are similar. But the angle  $OPD$  is a right angle by III. 31; therefore the angle  $OEQ$  is a right angle. Now  $E$  is a fixed point, therefore  $EQ$  is a fixed straight line. Hence the locus of  $Q$  is a straight line.

614. Let  $ABCD$  be a quadrilateral figure inscribed in a circle. Let  $BA$  and  $CD$  produced meet at  $P$ , and  $AD$  and  $BC$  produced, meet at  $Q$ . From  $B$  draw  $BE$  meeting  $PQ$  at  $E$  making the angle  $PBE$  equal to the angle  $PQD$ . Then the angle  $QBE$  is equal to the angle  $QPD$ : because the angles  $ADC$ ,  $ABC$  are together equal to two right angles by III. 22. Then the triangle  $PBE$  is similar to the triangle  $PQA$ , therefore the rectangle  $PA$ ,  $PB$  is equal to the rectangle  $PE$ ,  $PQ$  by VI. 4 and VI. 16. Similarly the rectangle  $QC$ ,  $QB$  is equal to the rectangle  $QE$ ,  $QP$ . Therefore by addition the rectangle  $PA$ ,  $PB$  together with the rectangle  $QC$ ,  $QB$  is equal to the rectangle  $PQ$ ,  $PE$  together with the rectangle  $QE$ ,  $PQ$ , that is to the square



on  $PQ$ . But the rectangle  $PA, PB$  is equal to the square on the tangent from  $P$ , and the rectangle  $QC, QB$  is equal to the square on the tangent from  $Q$  by III. 36.

615. Let  $H$  be the centre of the given circle,  $G$  the middle point of  $EF$ . Join  $HA, HE, HG, HF$ .

Then twice the square on  $EG$  together with twice the square on  $GH$  is equal to the sum of the squares on  $EH, FH$  by page 293 of the *Euclid*, that is to the rectangle  $EA, ED$  together with the rectangle  $FB, FA$ , together with twice the square on  $HA$ ; that is to the square on  $EF$ , together with twice the square on  $HA$  by Exercise 614, that is to four times the square on  $EG$  together with twice the square on  $HA$ . Therefore the square on  $GH$  is equal to the sum of the squares on  $HA, EG$ , and thus the radii drawn from  $H$  and  $G$  to a point of intersection of the two circles are at right angles by I. 48.

616. Let  $ABC$  be a right-angled triangle having the angle at  $B$  a right angle. Let  $BD$  be drawn from the angle at  $B$  perpendicular to  $AC$ : from  $D$  let  $DE, DF$  be drawn perpendicular to  $CB, BA$ . Join  $FE$ . Then  $FED$  is a triangle having the perpendiculars  $DF, DE$  as two of its sides. Also the angle  $EDF$  is a right angle since  $FDEB$  is a right-angled parallelogram.

Since  $D$  is on a circle of which  $BC$  is a diameter by III. 31,  $DE$  is not greater than one-half of  $BC$ . Similarly  $DF$  is not greater than one-half of  $AB$ .

617. Let  $PQ, RS$  be the two straight lines. Let them intersect at  $O$ . Let  $A$  be the middle point of  $RP, B$  of  $RS, C$  of  $QS, D$  of  $PQ$ . Then  $ABCD$  shall be a parallelogram. For  $AB$  and  $DC$  are both parallel to  $PS$  by Exercise 106. Therefore they are parallel to one another. Similarly  $AD$  and  $BC$  are parallel.

Now  $DC$  is half  $PS$  by Exercise 109; and by VI. 4 it may be shewn that the perpendicular from  $D$  on  $PS$  is half the perpendicular from  $Q$  on  $PS$ , and that the perpendicular from  $B$  on  $PS$  is half the perpendicular from  $R$  on  $PS$ . The parallelogram  $ABCD$  is equal to the difference of two parallelograms, each having  $DC$  for base, one having for height the perpendicular from  $B$  on  $PS$ , and the other having for height the perpendicular from  $D$  on  $PS$ . Thus the parallelogram  $ABCD$  is half the difference of two parallelograms each having  $DC$  for base, one having for height the perpendicular from  $R$  on  $PS$ , and the other having for height the perpendicular from  $Q$  on  $PS$ . Thus the parallelogram  $ABCD$  is half the difference of the triangles  $RPS$  and  $QPS$ , that is half the difference of the triangles  $ROP$  and  $QOS$ .

618. Let  $Q$  be the middle point of  $AB$ , and  $S$  the middle point of  $AC$ . Let  $O$  be the centre of the circle. Join  $AO$  cutting  $QS$  at  $P$ .

The difference of the squares on  $QO$  and  $OB$  is equal to the square on  $QB$  by I. 47, that is to the square on  $QA$ , that is to the squares on  $AP, PQ$ : therefore the difference of the squares on  $PO$  and  $OB$  is equal to the square on  $PA$ : therefore the difference of the squares on  $RO, OB$  is equal to the square on  $RA$ , that is the square on the tangent from  $R$  is equal to the square on  $RA$ : see III. 36. Therefore  $RA$  is equal to the tangent from  $R$ .

619. By the preceding Exercise the rectangle  $RQ, RP$  is equal to the square on  $RA$ , therefore  $RQ$  is to  $RA$  as  $RA$  is to  $RP$  by VI. 17; therefore

the triangle  $QRA$  is similar to the triangle  $ARP$  by VI. 6, therefore the angle  $AQR$  is equal to the angle  $RAP$ .

620. Let  $ABCD$  be the quadrilateral. Let  $a, b, c, d$  be the middle points of  $AB, BC, CD, DA$ , respectively; let  $E, F$  be the middle points of  $AC, DB$  respectively. Let  $O$  be the intersection of the circles round the triangles  $aFd, aEb$ ; we will prove that a circle will pass through  $dcEO$ .

The angle  $FOa$  is the difference of two right angles and the angle  $adF$ ; that is the difference of two right angles and the angle  $ABD$ . The angle  $aOE$  is the difference of two right angles and the angle  $abE$ , that is the difference of two right angles and the angle  $BAC$ . We shall obtain the angle  $doE$  by subtracting from four right angles, the excess of the angles  $aOE$ , and  $aOF$  above the angle  $FOd$ : thus the angle  $doE$  is equal to the sum of the angles  $ABD, BAC, FOd$ ; that is to the sum of the angles  $ABD, BAC, Fad$ ; that is to the sum of the angles  $ABD, BAC, ADB$ ; that is to the excess of two right angles and the angle  $BAC$  over the angle  $BAD$ ; that is to the difference of two right angles and the angle  $DAC$ ; that is to the difference of two right angles and the angle  $dcE$ . Therefore a circle can be described round  $dcEO$ , by page 276 of the *Euclid*.

Similarly a circle can be described round  $cFOb$ .

621. Let  $OA, OB, OC$  be the three straight lines which bisect the angles of an equilateral triangle. They meet at one point  $O$ , by page 312 of the *Euclid*. Let  $P$  be the given point from which  $PA, PB, PC$  are drawn perpendicular to  $OA, OB, OC$  respectively. Then the straight line  $PA$  shall be equal to the sum of the straight lines  $PB$  and  $PC$ , supposing  $PA$  to be the longest perpendicular. Draw  $PD$  parallel to  $OB$  meeting  $OC$  at  $D$ . Draw  $DH$  from  $D$  perpendicular to  $OA$ , and  $DF$  perpendicular to  $PA$ . Then  $DH$  is equal to  $BP$  because  $OC$  bisects the angle  $BOH$ .

From the triangles  $PCD, PFD, PC$  is equal to  $PF$ . Also  $BP$  has been shewn to be equal to  $DH$ , that is to  $AF$ . But  $AF, FP$  are together equal to  $AP$ , therefore  $BP, PC$  are together equal to  $AP$ .

622. Draw the straight line  $AF$  bisecting the exterior angle between  $AC$  and  $AD$ , and meeting the circle  $ABC$  at  $E$ ; and draw the tangents  $TE, TF$  meeting at  $T$ . Then the angles  $EAC, FAD$  are equal; therefore  $FBD, EBC$  are equal by III. 21. Therefore the complements of these angles are equal, that is  $ABF$  is equal to  $ABE$ , and thus by III. 32 the angles  $TEF$  and  $TFE$  are equal. Therefore  $T$  is on the line  $AB$  produced; see Exercise 527.

If the interior angle between  $AC$  and  $AD$  be bisected the proof is substantially the same.

623. Let  $ABC$  be the given triangle. Through  $E$  and  $D$  the middle points of  $AC$  and  $AB$  respectively draw parallel straight lines  $EF, DG$  meeting  $BC$  at the given angle. Through  $A$  draw a straight line parallel to  $BC$  and let  $GD, FE$  produced meet this straight line at  $K$  and  $H$  respectively. Then  $FGKH$  is the parallelogram required; for it is obvious that the triangle  $AEH$  is equal to the triangle  $FEC$ , and that the triangle  $KAD$  is equal to the triangle  $BDG$ .

624. Draw the diagonals  $AC$  and  $BD$  intersecting at  $O$ . Suppose  $P$  to be within the triangle  $DOC$ .

We have to shew that the triangle  $PAC$  is equal to the difference of the triangles  $PAB$  and  $PAD$ ; this we will do by shewing that the sum of the triangles  $PAD$  and  $PAC$  is equal to the triangle  $PAB$ .

The sum of the triangles  $PAD$  and  $PAC$  is equal to the sum of the triangles  $POD$ ,  $DOA$ ,  $POC$ . The triangle  $PAB$  is equal to the sum of the triangles  $POB$ ,  $BOA$ ,  $POA$ . Now  $AC$  and  $BD$  bisect each other by Exercise 78, therefore by I. 38 the triangle  $POD$  is equal to the triangle  $POB$ ; the triangle  $DOA$  is equal to the triangle  $BOA$  and the triangle  $POC$  is equal to the triangle  $POA$ . Thus the required result is obtained.

In a similar way any other case can be treated.

625. Let  $R$ ,  $Q$  be the two points where the circles cut one another.

The rectangle  $AC$ ,  $CD$  is equal to the rectangle  $RC$ ,  $CQ$ , that is to the rectangle  $EC$ ,  $CB$  by III. 35; therefore  $AC$  is to  $CE$  as  $CB$  is to  $CD$  by VI. 16; therefore by V. 18  $AE$  is to  $CE$  as  $BD$  is to  $CD$ : therefore  $AE$  is to  $BD$  as  $CE$  is to  $CD$  by V. 16. Similarly  $AE$  is to  $BD$  as  $CA$  is to  $CB$ ; therefore the square on  $AE$  is to the square on  $BD$  as the rectangle  $AC$ ,  $CE$  is to the rectangle  $BC$ ,  $CD$ , therefore the square on  $BD$  is to the square on  $AE$  as the rectangle  $BC$ ,  $CD$  is to the rectangle  $AC$ ,  $CE$ .



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